Method of Images and the Sphere Green Function

- Method of Images
- Point Charge in the Presence of a Grounded Conducting Sphere
- Point Charge in the Presence of a Charged Insulated Conducting Sphere
- Point Charge in the Presence of a Conducting Sphere at Fixed Potential
- Green Function for the Sphere

(Scanned class notes – S. F. Hassan)

Boundary Value Problem in Electrostatics

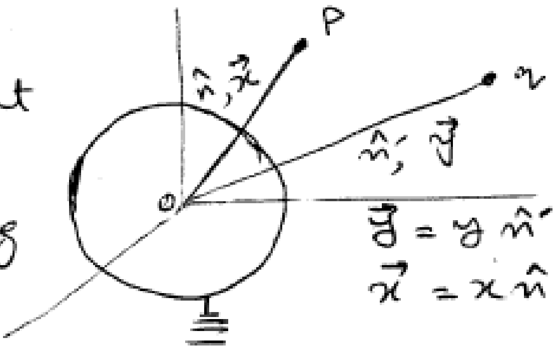
Our intention is to develop some general techniques to solve the Poisson or Laplace equations subject to appropriate boundary conditions, and probably illustrate them in some cases. We will not discuss the determination of fields for various standard charge configurations which are treated in detail in many text books.

Method of Images

Suppose we are interested in determining the fields in a volume V. In some situations it is possible to replace the boundary conditions on the boundary of V, by a few charges external to V. These charges then simulate the boundary conditions on $S = \partial V$. Inside **V** the two arrangements are equivalent and one can solve the image charge problem. The image charges must be external to V or else they will not satisfy the Laplace eqn. in V. Outside V the problem with image charges is not equivalent to the original boundary value problem. A simple example is a point charge in front of an infinite plane conductor at zero potential.

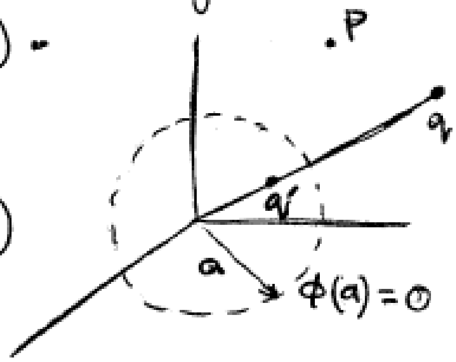
Point charge in the presence of a grounded conducting sphere :

Setup :



consider a point charge at $\vec{y} = y\,\hat{n}'$ outside a grounded ($\Phi = 0$) conducting sphere of radius a. We want to determine the potential at point P at $\vec{x} = x\hat{n}$, also outside the sphere (Question: why is the potential inside the sphere zero?).

This corresponds to solving



$$\nabla^2 \Phi(\vec{x}) = -4\pi q\, \delta(\vec{x} - \vec{y})$$

$$\Phi(|\vec{x}| = a) = 0$$

(note that for $q = 1$, Φ is the same as the Green function for Dirichlet problem). Instead of directly solving these equations, we construct the soln using the method of images.

The Image Problem :

We put an image point q' inside the sphere such that q and q' give rise to $\Phi(|x| = a) = 0$. The boundary problem has rotational symmetry around the axis $\vec{y}$ which should be preserved by the image charge. Thus q' is located on $\vec{y}$ say at $\vec{y}' = y'\,\hat{n}'$.

Then, $\quad \Phi(\vec{x}) = \dfrac{q}{|\vec{x}-\vec{y}|} + \dfrac{q'}{|\vec{x}-\vec{y}'|}$

q' and $\vec{y}'$ are determined by

$$\Phi(a) = \frac{q}{|a\hat{n} - y\hat{n}'|} + \frac{q'}{|a\hat{n} - y'\hat{n}'|} = 0. \quad \underline{(\Rightarrow qq' < 0)}$$

$$\Rightarrow \frac{q|a\hat{n} - y'\hat{n}'| + q'|a\hat{n} - y\hat{n}'|}{|a\hat{n} - y\hat{n}'||a\hat{n} - y'\hat{n}'|} = 0$$

$$\Rightarrow q^2(a^2 + y'^2 - 2ay'\cos\gamma) = q'^2(a^2 + y^2 - 2ay\cos\gamma)$$

$$\underline{\text{for any } \hat{n}' \cdot n = \cos\gamma.}$$

$$\Rightarrow q^2 y' = q'^2 y, \quad q^2(a^2 + y'^2) = q'^2(a^2 + y^2)$$

substituting for q'^2 in the second eqn.,

$$y'^2 - y'\left(\frac{a^2}{y} + y\right) + a^2 = 0 \Rightarrow y' = \frac{a^2}{y}$$

$$= \cancel{y}$$

$$\therefore \boxed{q' = -\frac{a}{y} q}, \quad \boxed{y' = \frac{a^2}{y}}$$

$$\boxed{\Phi(x) = \frac{q}{|x\hat{n} - y\hat{n}'|} - \frac{aq}{y} \frac{1}{|x\hat{n} - \frac{a^2}{y}\hat{n}'|}} \quad (x \geq a)$$

For $\vec{x}$ inside the sphere (where the method of images cannot be used), $\phi(x)=0$:
Here $\phi(x)$ is determined by $\nabla^2\phi=0$ and the b.c. $\phi|_{x=a}=0$, independent of the presence of q outside. Hence the solution is the same as for $q=0$.

Surface charge density:
When $q=0$, the grounded conductor carries zero charge. In the presence of q, it soaks up charge from ground and distributes it over its surface to cancel the potential due to q. Gauss law gives this surface charge density as

$$(\vec{E}_2-\vec{E}_1)\cdot\hat{n}=4\pi\sigma$$

$\vec{E}_2$ is on the same side as $\hat{n}$

$\hat{n}$ is a unit vector pointing "outwards", so $\hat{n}=-\hat{r}$ (or $-\hat{x}$). Also $\vec{E}_2=$ field in the outside region (direction of $\hat{n}$) $=0$. So that

$$\vec{E}_1\cdot\hat{r}=4\pi\sigma. \quad \text{But}, \quad \vec{E}_1\cdot\hat{r}=-\nabla\phi\cdot\hat{r}=\frac{-\partial\varphi}{\partial x}.$$

$$\boxed{\sigma=-\frac{1}{4\pi}\frac{\partial\varphi}{\partial x}\bigg|_{x=a}}$$

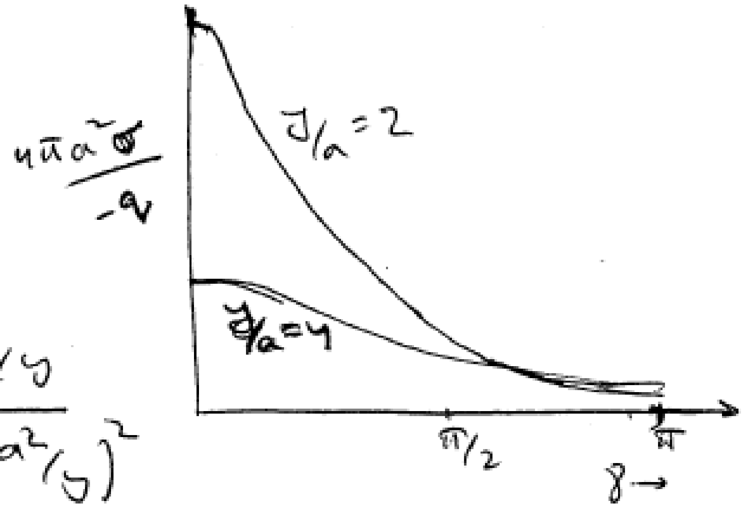
$$\frac{\partial\varphi}{\partial x}=-\frac{q(x-y\cos\gamma)}{(x^2+y^2-2xy\cos\gamma)^{3/2}}-\frac{q'(x-y'\cos\gamma)}{(x^2+y'^2-2xy'\cos\gamma)^{3/2}}$$

$$\boxed{\left.\frac{\partial \varphi}{\partial x}\right|_{x=a} = \frac{q}{ya}\frac{\left(1-a^2/y^2\right)}{\left(1+a^2/y^2-\frac{2a}{y}\cos\gamma\right)^{3/2}}}$$

$$\boxed{\sigma = -\frac{1}{4\pi}\frac{q}{a^2}\left(\frac{a}{y}\right)\frac{\left(1-a^2/y^2\right)}{\left(1+a^2/y^2-2\,a/y\cos\gamma\right)^{3/2}}}$$

Gauss law on a surface enclosing the sphere will give the total charge within as q'. Therefore,

$$q' = \oint_{S(a)} \sigma\, ds$$ as can be verified.



Force on q:

$$|\vec{F}| = \frac{qq'}{|y-y'|^2} = \frac{q^2\, a/y}{\left(y-a^2/y\right)^2}$$

$$\boxed{|\vec{F}| = \frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1-\frac{a^2}{y^2}\right)^{-2}}$$ attractive at all distances $y \geq a$

At large distances $\frac{a}{y} \rightarrow 0$, $|\vec{F}| \sim \left(\frac{a}{y}\right)^3 \rightarrow$ dipole at the centre. ($q' = \frac{a}{y} q \rightarrow 0$).

The procedure is the same for q inside the sphere and q' outside the sphere

Point charge in the presence of a charged, insulated conducting sphere:

The setup is as in the previous example except that the conducting sphere is insulated and carries total charge Q.

The potential at $\vec{x}$ is independent of how charge Q is transferred to the sphere. We can then split $Q = q' + (Q - q')$ and use the superposition principle.

Start with the grounded conductor of the previous example. It carries charge q' distributed such that $\Phi(a) = 0$. Now, insulate the sphere and add charge $Q - q'$ to it. Since the surface was at zero potential, this additional charge distributes itself uniformly over the sphere. Outside the sphere, this is equivalent to a point charge $Q - q'$ at the centre. Therefore, the image charges are q' at $\vec{y}'$ and $(Q - q')$ at the origin.

$$\Phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|} + \frac{Q-q'}{|\vec{x}|}$$

Using $\quad q' = -\frac{a}{y} q \quad , \quad y' = \frac{a^2}{y}$,

$$\boxed{\phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y\,|\vec{x}-\frac{a^2}{y^2}\vec{y}|} + \frac{Q+\frac{a}{y}q}{|\vec{x}|}}$$

Force on q:

$$\vec{F}_q = \frac{qq'}{|\vec{y}-\vec{y}'|^2}\hat{y} + \frac{q(Q-q')}{|\vec{y}|^2}\hat{y}$$

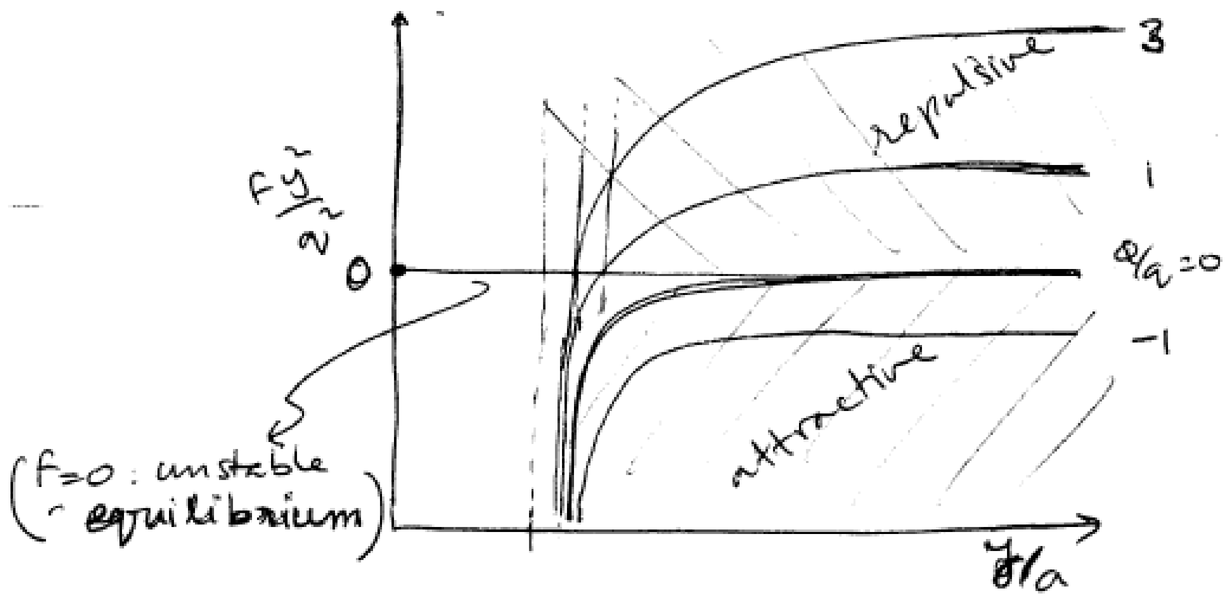
$$= q\left(\frac{Q}{y^2} + \frac{aq}{y^3} - \frac{ayq}{(y^2-a^2)^2} \right)\hat{y}$$

$$= \frac{q}{y^2}\left(Q + \frac{a^5 q - 2a^3y^2 q}{y(y^2-a^2)^2} \right)\hat{y}$$

$$\boxed{\vec{F}_q = \frac{q}{y^2}\left(Q - \frac{a^3 q(2y^2-a^2)}{y(y^2-a^2)^2} \right)\frac{\vec{y}}{y}}$$

* for $y \to \infty$, $F_q \to qQ/y^2$, as expected.
* For $Q \leqslant 0$, this force on q is always attractive ($q>0$).

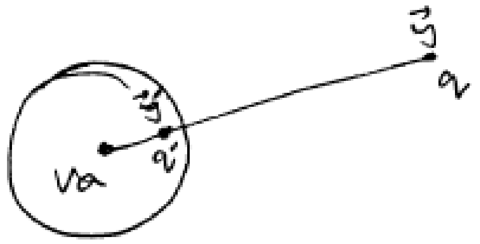
For $Q>0$, the force is repulsive at large distances (as expected). However, at small enough distances, the second term dominates and the force becomes attractive ($y-a\to 0$). When q is very close to the surface of the sphere, it experiances a very large attractive force even when both carry charges of the same sign. This is the reason why elementary charges do not leave the surface of a charged conductor: As soon as a point charge is moved away from the surface of the conductor, an attractive force is developed due to the induced charge on the surface.



Point charge in the presence of a conducting sphere at fixed potential (V).

The field $\Phi(\vec{x})$ is independent of how the sphere is raised to the fixed potential V. Start with the grounded sphere of the first example, $\Phi(a)=0$. This corresponds to an image charge q' at $\vec{y}'$. Raising the potential to $\Phi(a)=V$ is equivalent to putting an image charge aV at the centre.

$$\Phi(x) = \frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|} + \frac{Va}{|\vec{x}|}$$



This is related to the previous example by

$$Q - q' = Va \quad \Rightarrow \quad Q = Va + q' = Va - \frac{qa}{y}.$$

Using this, the force on q can be obtained as

$$\boxed{\vec{F}_q = \frac{aq}{y^2}\left(V - \frac{q y^3}{(y^2-a^2)^2} \right) \hat{y}}$$

Again, F_q is attractive for $\frac{a}{y} \to 1$ and the unstable equilibrium point is closer to the surface for larger V.

Green Function for the Sphere, General Boundary conditions:

We can now consider a more general problem: This is a sphere with a general potential $\phi(a,\theta',\phi')$ specified on the surface and some given charge distribution outside.

$$\nabla^2 \phi(\vec{x}) = -4\pi\rho(\vec{x}) \qquad (|\vec{x}|>a)$$

$$\phi(|\vec{x}|=a) = \phi(a,\theta',\phi')$$

This defines a Dirichlet problem with soln:

$$\phi(x) = \int_V \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \phi(x')\,\frac{\partial G_D}{\partial n'}\, ds'$$

where

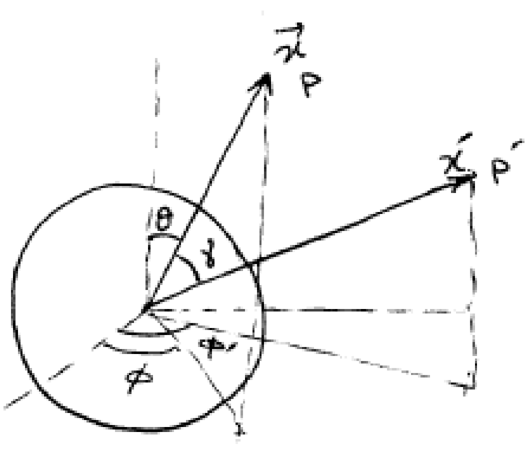
$$\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}').$$

$$G_D(\vec{x},\vec{x}')\Big|_{|\vec{x}'|=a} = 0.$$

Clearly, $G_D(\vec{x},\vec{x}')$ is the solution for the potential at $\vec{x}'$ due to a unit charge at $\vec{x}$ in the presence of a grounded conducting sphere. This was the first example we solved by the method of images (In $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$,

$F(\vec{x},\vec{x}')$ can be regarded as the potential due to the image charge inside the sphere).

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{q'}{|\vec{x}-\vec{y}'|}$$



$$q' = -\frac{a}{x'}, \quad y' = \frac{a^2}{x'}$$

$\Rightarrow$

$$\boxed{G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right|}}$$

$\vec{x} = (x, \theta, \varphi)$
$\vec{x}' = (x', \theta', \varphi')$

In spherical coordinates,

$$\boxed{G_D(\vec{x},\vec{x}') = \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{(x'^2x^2+a^2-2xx'\cos\gamma)^{1/2}}}$$

This is symmetric **under** $\vec{x}, \vec{x}'$ interchange and $G_D(x, |x'|=a) = 0$.

The soln, Φ involves $\partial G_D/\partial n'$ $(\hat{n} = -\hat{x}')$:

$$\boxed{\left.\frac{\partial G_D}{\partial n'}\right|_{x'=a} = -\frac{(x^2-a^2)}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}}$$

$\therefore$ for $\rho = 0$, the soln. of the Laplace eqn. outside the sphere, with potential specified on the surface, is

$$\boxed{\Phi(x) = \frac{1}{4\pi}\int \Phi(a,\theta',\varphi')\frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}\, a^2 d\Omega'}$$

where, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

Note that the position $\vec{x}$ of the unit charge in the Green function problem corresponds to the observation point in the original boundary value problem. A charge distribution $\rho(\vec{x})$, if it is non-zero in the original problem, has no effect on the Green function.