Rapid Review of Vector Calculus

- Gradient
- Divergence
- Curl
- Some Useful Results

(Class notes – Fawad Hassan)

1 Rapid Review of Vector Calculus

In this section we use the following notation:

$$\begin{aligned} \text{Position vector} &: \ \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \text{Scalar field} &: \ \Phi(\vec{r}) \equiv \Phi(x,y,z) \\ \text{Vector field} &: \ \vec{A}(\vec{r}) = A_x(\vec{r})\hat{\imath} + A_y(\vec{r})\hat{\jmath} + A_z(\vec{r})\hat{k} \\ \text{“Del” operator} &: \ \vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \end{aligned}$$

The study of spatial variations of scalar and vetor fields involves vector calculus. One can define three types of differentiations: *gradient*, *divergence* and *curl*. These are described below without giving proofs.

1.1 Gradient ($grad\Phi, \vec{\nabla}\Phi$):

This operation converts a scalar field Φ into a vector field $\vec{\nabla}\Phi$. It can be defined as

$$\boxed{\vec{\nabla}\Phi = \hat{\imath}\frac{\partial}{\partial x}\Phi + \hat{\jmath}\frac{\partial}{\partial y}\Phi + \hat{k}\frac{\partial}{\partial z}\Phi} \tag{1}$$

The meaning of $\vec{\nabla}\Phi$: Consider Φ at two closely separated points in space; $\Phi(\vec{r})$ and $\Phi(\vec{r}+\vec{\Delta l})$. Then,

$$\frac{\partial\Phi}{\partial l} = \lim_{|\Delta l|\to 0} \frac{\Phi(\vec{r}+\vec{\Delta l}) - \Phi(\vec{r})}{|\Delta l|}$$

is the *directional derivative* of Φ along $\vec{\Delta l}$ at $\vec{r}$. Its value depends not only on the point $\vec{r}$, but also on the direction of $\vec{\Delta l} = \hat{\imath}\Delta x + \hat{\jmath}\Delta y + \hat{k}\Delta z$. This can be evaluated by doing a simple Taylor expansion to first order,

$$\Phi(\vec{r}+\vec{\Delta l}) \equiv \Phi(x+\Delta x, y+\Delta y, z+\Delta z) = \Phi(x,y,z) + \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z \cdots$$

and noting that, in terms of the *scalar product* of $\vec{\nabla}\Phi$ and $\vec{\Delta l}$, it can be written as

$$\Phi(\vec{r}+\vec{\Delta l}) = \Phi(\vec{r}) + \vec{\nabla}\Phi\cdot\vec{\Delta l} + \cdots$$

Using this in the equation for $\partial\Phi/\partial l$ and definig $\lim_{|\Delta l|\to 0}\frac{\vec{\Delta l}}{|\Delta l|} = \hat{\Delta l}$ as the unit vector along $\vec{\Delta l}$, we have a relation between the gradient of Φ and its directional derivative,

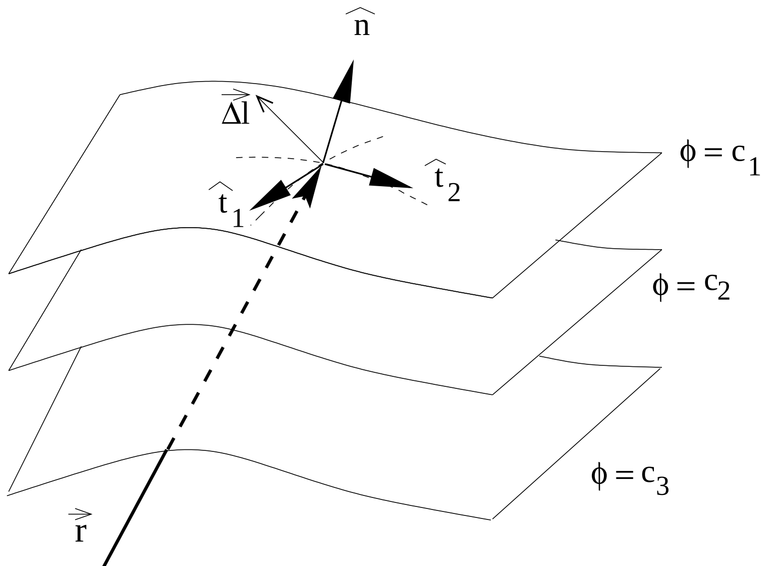
$$\boxed{\frac{\partial\Phi}{\partial l} = \vec{\nabla}\Phi\cdot\hat{\Delta l}} \tag{2}$$

This equation enables us to easily compute directional derivatives.

As a vector, $\vec{\nabla}\Phi$ is specified by its *magnitude* $|\vec{\nabla}\Phi|$, and its *direction* [1]. To visualize these, note that, in general, through any point $\vec{r}$ in space, one can draw a 2-dimensional surface over which Φ is constant. This is because the condition $\Phi(x, y, z) = c$ can be regarded as a constraint on the 3 coordinates and can be solved, for example, for $z = f(x, y; c)$ which is the equation for a 2-dimensional surface in space. Hence, in general, space is filled with surfaces of constant Φ, for various values of the constant c (see figure).

Now, consider a constant Φ surface through $\vec{r}$. Clearly for $\hat{\Delta l}$ tangential to this surface, the directional derivative of Φ is zero. From the right hand side of (2) it then follows that the vector $\vec{\nabla}\Phi$ is perpendicular to the constant-Φ surface at $\vec{r}$. In other words, if $\hat{n}$ denotes a unit normal to the surface at $\vec{r}$, then,

$$\vec{\nabla}\Phi = |\vec{\nabla}\Phi|\,\hat{n}$$



Surfaces of constant ϕ (x,y,z)

Let us now compute the directional derivative in directions perpendicular to the surface of constant Φ

$$\frac{\partial \Phi}{\partial l}\Big|_{\hat{\Delta l}=\hat{n}} \equiv \frac{\partial \Phi}{\partial n} = |\vec{\nabla}\Phi|$$

Since $\hat{n}\cdot\hat{n} = 1$. Hence the magnitude of $\vec{\nabla}\Phi$ is the value of the directional derivative when $\hat{\Delta l}$ is normal to the constant Φ surface. This also is the *maximum* value of the directional derivative, since for any other direction $\hat{\Delta l}$ making an angle θ with $\hat{n}$, one has

$$\frac{\partial \Phi}{\partial l} = |\vec{\nabla}\Phi|\,(\hat{n}\cdot\hat{\Delta l})$$

and $|\hat{n}\cdot\hat{\Delta l}| = |cos\theta| \leq 1$.

To summarize, $\vec{\nabla}\Phi$ is the directional derivative of Φ in the direction of the normal to the constant-Φ surface at $\vec{r}$. Its magnitude is the maximum value of the directional derivative at $\vec{r}$. All other values of the directional derivative are obtained from this maximum value through (2).

Let us now consider a curve l from $\vec{r}_1$ to $\vec{r}_2$ and let $\vec{dl}$ be a small displacement vector along this curve. Note that $\int_{\vec{r}_1}^{\vec{r}_2}\vec{\nabla}\Phi\cdot\vec{dl} = \int_{\vec{r}_1}^{\vec{r}_2}(\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz) = \int_{\vec{r}_1}^{\vec{r}_2} d\Phi$, so that we

[1] Recall that a vector $\vec{V}$ has a magnitude $V = \sqrt{\vec{V}\cdot\vec{V}}$ and a direction $\hat{V} = \vec{V}/V$ in terms of which it is expressed as $\vec{V} = V\hat{V}$.

have the line integral result,

$$\boxed{\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla}\Phi \cdot \vec{dl} = \Phi(\vec{r}_2) - \Phi(\vec{r}_2)} \tag{3}$$

In particular, the integral depends only on the end points $\vec{r}_1$ and $\vec{r}_2$ of the curve, not on its shape, and hence is *path independent.* Equivalently, it vanishes for a closed path, $\oint \vec{\nabla}\Phi \cdot \vec{dl} = 0$.

1.2 Divergence $(div\vec{A}, \vec{\nabla}\cdot\vec{A})$

This operation converts a vector field $\vec{A}$ into a scalar field $\vec{\nabla}\cdot\vec{A}$. It can be defined as

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z} \tag{4}$$

which is simply the *scalar product* of $\vec{\nabla}$ and $\vec{A}$.

The meaning of $\vec{\nabla}\cdot\vec{A}$: Consider a very small surface element $\Delta\vec{S} = \Delta S\,\hat{n}$, where $\hat{n}$ is a unit vector normal to the surface. The flux of a vector field $\vec{A}(\vec{r})$ across this surface (over which its variation can be neglected) is defined as $\vec{A}\cdot\Delta\vec{S}$. For example, in a liquid flow with local density $\rho(\vec{r})$ and velocity field $\vec{v}(\vec{r})$, $\vec{A} = \rho(\vec{r})\vec{v}(\vec{r})$ is the current density and $\vec{A}\cdot\Delta\vec{S}$ gives the liquid flow across ΔS per unit time. Now consider a volume ΔV bounded by a surface S. The total flux of $\vec{A}$ across the closed surface is

$$\oint_S \vec{A}\cdot\vec{ds}$$

Since S surrounds ΔV, this is obviously the flux leaving or entering the volume element. Then for a small ΔV around $\vec{r}$, as $\Delta V \to 0$ the flux per unit volume becomes a local property of the vector field at $\vec{r}$, defined as its divergence,

$$\lim_{\Delta V\to 0} \frac{\oint_S \vec{A}\cdot\vec{ds}}{\Delta V} = \vec{\nabla}\cdot\vec{A}(\vec{r})$$

It can be shown that this definition is quivalent to the one in equation (4). The identification of $\vec{\nabla}\cdot\vec{A}$ as flux per unit volume allows us to write two equivalent expressions for the total flux leaving or entering a finit volume V,

$$\boxed{\int_V \vec{\nabla}\cdot\vec{A}dv = \oint_S \vec{A}\cdot\vec{ds}} \tag{5}$$

This volume integral result is called the *divergence theorm* which relates a volume integral to a surface integral.

1.3 Curl ($curl\vec{A}, \vec{\nabla}\times\vec{A}$):

This operation converts a vector field $\vec{A}$ into another vector field $\vec{\nabla}\times\vec{A}$. It is defined as the *vector product* of $\vec{\nabla}$ and $\vec{A}$,

$$\boxed{\vec{\nabla}\times\vec{A} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{pmatrix}} \qquad (6)$$

Or more explicitly,

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

The meaning of $\vec{\nabla}\times\vec{A}$: Consider a small line element $\vec{dl}$ on a closed loop l that forms the boundary of a surface ΔS. For a vector field $\vec{A}(\vec{r})$, one can evaluate the closed line integral

$$\oint_l \vec{A}\cdot\vec{dl}$$

For example, if $\vec{A}$ denotes the force acting on a particle, this integral evaluates the total work done in taking the particle around the closed loop l. If ΔS is shrunk to zero size around the point $\vec{r}$, then the line integral essentially becomes a property of the vector field $\vec{A}$ at $\vec{r}$. Since ΔS also has an orientation in space, specified by the unit normal $\hat{n}$, this property depends not only on the point $\vec{r}$, but also on the orientation $\hat{n}$. $\vec{\nabla}\times A$ can also be defined in term of this property of the vector field,

$$\lim_{\Delta S\to 0}\frac{\oint_l \vec{A}\cdot\vec{dl}}{\Delta S} = (\vec{\nabla}\times\vec{A})\cdot\hat{n}$$

Then, for a finite area S this leads to the integral result,

$$\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \oint_l \vec{A}\cdot\vec{dl}} \qquad (7)$$

This is called *Stokes's theorem* and relates an integral over a surface to a closed line integral over its boundary.

1.4 Some Useful Results

i) One can easily verify that $\vec{\nabla}\times(\vec{\nabla}\Phi) = 0$ by using the definitions of curl and gradient given above in the rectangular coordinate system. Alternatively, set $\vec{A} = \vec{\nabla}\Phi$ in Stokes's theorem. Then from equation (3) it follows that the right hand side vanishes. Since this is true for any choice of the surface S on the left hand side, one concludes that

$$\vec{A} = \vec{\nabla}\Phi \Rightarrow \vec{\nabla}\times\vec{A} = 0$$

The converse also holds,

$$\vec{\nabla}\times\vec{A} = 0 \Rightarrow \vec{A} = \vec{\nabla}\Phi$$

and plays in important role in electrostatics. To prove this, note that by Stokes's theorem $\vec{\nabla}\times\vec{A} = 0$ implies $\oint_l \vec{A}\cdot\vec{dl} = 0$ for a closed loop l. This in turn implies the path independence of $\int_{\vec{r}_A}^{\vec{r}_B} \vec{A}\cdot\vec{dl}$ (To see this, break up l as $l_{AB} + l'_{BA}$. Varying l and keeping l' fixed, one concludes that the intgral over l should only depend on its end points). Hence,

$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{A}\cdot\vec{dl} = \Phi(\vec{r}_A, \vec{r}_B)$$

Now, varying $\vec{r}_B$ along some $\vec{\Delta l}$, the left and the right hand sides vary separately as

$$\delta(LHS) = \int_{\vec{r}_A}^{\vec{r}_B+\vec{\Delta l}} \vec{A}\cdot\vec{dl} - \int_{\vec{r}_A}^{\vec{r}_B} \vec{A}\cdot\vec{dl} = \vec{A}\cdot\vec{\Delta l}|_B = \vec{\nabla}\Phi\cdot\vec{\Delta l}|_B = \delta(RHS)$$

To find the variation of the right hand side, we have used equation (2). Since the equality holds for any $\vec{\Delta l}$, it leads to $\vec{A} = \vec{\nabla}\Phi$, as desired.

ii) For $r = \sqrt{x^2+y^2+z^2}$, one can easily show, by direct calculation, that $\vec{\nabla}\cdot\vec{r} = 3$, $\vec{\nabla}r = \hat{r}$ and

$$\vec{\nabla}(\frac{1}{r}) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

More generally, shifting the origin to $\vec{r'}$, one gets

$$\boxed{\vec{\nabla}(\frac{1}{|\vec{r}-\vec{r'}|}) = -\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}} \tag{8}$$

iii) Let us now evaluate $\nabla^2(\frac{1}{r})$, where $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$ is called the Laplacian operator. For $r \neq 0$, direct computation yields,

$$\nabla^2\left(\frac{1}{r}\right) \equiv \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = -\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \cdots\right] = 0$$

The only place where this need not hold is the point $r = 0$. To see if there is any contribution from $r = 0$, we integrate $\nabla^2(\frac{1}{r})$ over a volume in space that contains this point. For convenience, let us choose a spherical volume $V(R)$ of radius R and surface area $S(R) = 4\pi R^2$. Then using the divergence theorem (4), one gets

$$\int_{V(R)} \nabla^2\left(\frac{1}{r}\right) dv = \oint_{S(R)} \vec{\nabla}\left(\frac{1}{r}\right)\Big|_{r=R} \cdot \vec{ds} = -\frac{1}{R^2}\oint_{S(R)} \hat{R}\cdot\vec{ds} = -4\pi$$

In the last step we have used the fact that since $\hat{R}$ is a unit radial vector, $\hat{R}\cdot\vec{ds} = ds$ and the integral computes the area of the sphere which is $4\pi R^2$. This shows that the integrand is zero everywhere except for $r = 0$ and therefore is a delta function supported at this point,

$$\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r})$$

Again, shifting the origin to some $\vec{r'}$, this result takes the form

$$\boxed{\nabla^2(\frac{1}{|\vec{r} - \vec{r'}|}) = -4\pi\delta(\vec{r} - \vec{r'})} \tag{9}$$