From Coulom's Law to Green Funtions

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(Scanned class notes – S. F. Hassan)

Notes on Electrostatics

Overview: The study of electrostatics is based on the two time independent Maxwell equations.

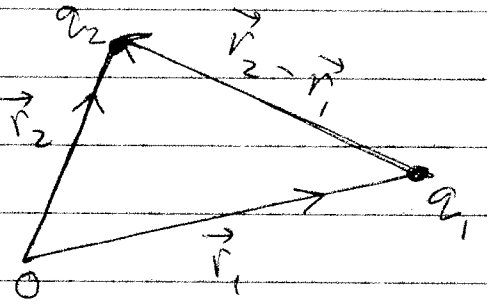
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho, \qquad \vec{\nabla}\times\vec{E} = 0.$$

Below we will describe the 1) origin, 2) meaning and 3) solutions of these equations.

Coulomb's Law:

This gives the force between two point charges based on experimental observation. Consider charges q_1 and q_2 at positions $\vec{r}_1$ and $\vec{r}_2$.

They are seperated by $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$ that points from q_1 to q_2. Then, the force acting on q_2 due to q_1 is given by



$$\vec{F}_{21} = k q_1 q_2 \frac{\vec{r}_{21}}{|\vec{r}_{21}|^3} = k q_1 q_2 \frac{\hat{r}_{21}}{|r_{21}|^2}$$

Clearly, for the force acting on q_1 due to q_2 we have $\vec{F}_{12} = -\vec{F}_{21}$ consistent with Newton's third law. k is a proportionality constant the value of which depends on the system of units employed.

Units

1) In the MKSA system of units (also called the SI system, the unit of "F" is Newtons, that of "r" is meters and charge "q" is measured in coulombs (which is in turn defined in terms of the unit of electric current). Then "k" is the only unknown in the Coulomb law and its value can be determined experimentally by measuring the force between two known charges. In vacuum it is usually written as

$$k = \frac{1}{4\pi \varepsilon_0}$$

where, $\varepsilon_0 = 8.854 \times 10^{-12}$ farad/meter is called the permittivity of free space. The value of k also depends on the nature of the medium between the two charges as will be discussed later.

2) Another possibility is not to fix the unit of charge before hand but define it through the Coulomb force law. The freedom in choosing the unit of charge can be used to set $k = 1$. Then the charge is said to be 1 statcoulomb if for $q_1 = q_2$ and $r = 1$ cm, one gets $F = 1$ dyne. This is the electrostatic unit (esu) system. We will work mostly in this unit system, although for the most part we will retain k in the formulas so that the SI units can also be easily recovered.

Superposition Principle :

Some features of the Coulomb law are obvious from the formula :

(i) $F_{12} \propto q_1 q_2$

(ii) $F_{12} \propto \frac{1}{r_{12}^2}$

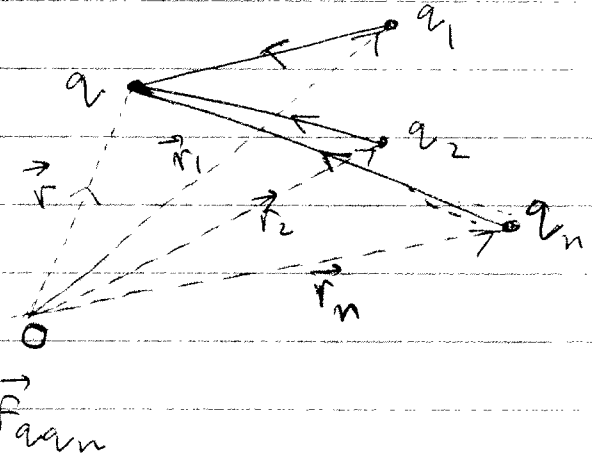
(iii) $\vec{F}_{21}$ is directed along $\vec{r}_{21}$. Such forces are said to be "central"; gravity being another example

(iv) The force is repulsive ($|\vec{F}| > 0$) for charges of the same sign and attractive ($|\vec{F}| < 0$) for opposite charges. (here $|\vec{F}|$ denotes the vector modulus of $\vec{F}$ and not its absolute value).

There is another property that is not obvious from the force equation but follows from experiments. This is the <u>superposition property</u> which states that the Coulomb force between two charges is not affected by the presence of other charges.

Thus the total force on a charge q due to charges $q_1, q_2 \ldots q_n$ is given by

$$\vec{F}_q(q_1 \ldots q_n) = \vec{F}_{qq_1} + \vec{F}_{qq_2} \ldots \vec{F}_{qq_n}$$

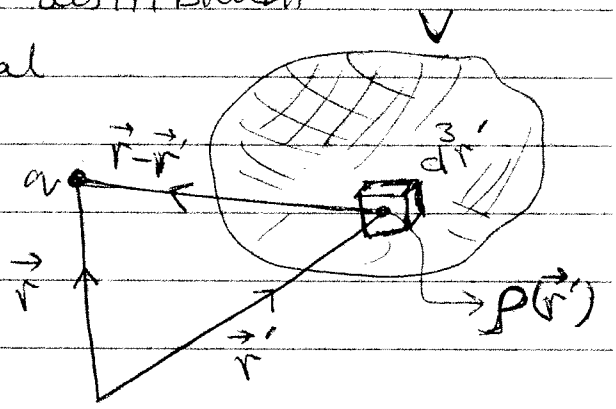


or

$$\boxed{\vec{F}_q(q_1 \ldots q_n) = kq \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i)}$$

Electric Field and the Generalized Coulomb Law

Consider a continuous charge distribution within a volume V and local charge density $\rho(\vec{r}')$. The force on q due to the charge within a small volume element d^3r' around the point $\vec{r}'$ is



$$\Delta \vec{F}_q = k \frac{q(\rho(\vec{r}')d^3r')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

Hence the total force is given by the volume integral,

$$\vec{F}_q = kq \int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\, d^3r'$$

Note that $\vec{F}_q/q$ is a property of the charge distribution $\rho(\vec{r}')$ alone and we can regard q as a test charge used to probe this property of $\rho(\vec{r}')$. In practice, q should be small enough that its presence at $\vec{r}$ does not distort $\rho(\vec{r}')$ appreciably (or else q cannot be used as a probe). Hence we define the Electric Field $\vec{E}(\vec{r})$, as a property of the charge distribution $\rho(\vec{r}')$ measured by a probe charge at $\vec{r}$, to be given by

$$\vec{E}(\vec{r}) = \lim_{q \to 0} \vec{F}_q / q \qquad \text{or} \qquad \vec{F}_q = q\,\vec{E}(\vec{r}).$$

Explicitly:

$$\boxed{\vec{E}(\vec{r}) = k \int \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d^3 r'}$$

This is the generalized Coulomb law. A discrete set of charges q_i at positions $\vec{r}_i$ can be described in terms of a $\rho(\vec{r}')$ by choosing

$$\rho(\vec{r}') = \sum_{i=1}^{n} q_i \, \delta(\vec{r}' - \vec{r}_i)$$

Then,

$$\vec{E}(\vec{r}) = k \sum_{i=1}^{n} q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

The total charge becomes

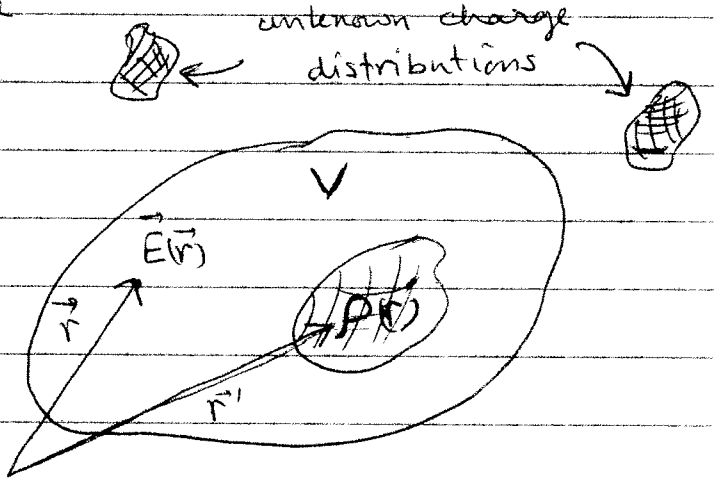
$$\int_V d^3 r \, \rho(r) = \sum_{i=1}^{n} q_i$$

For a single charge one gets $\vec{E}(\vec{r}) = k q_1 \frac{\vec{r} - \vec{r}_1}{|r - r_1|^3}$ which must be familiar from elementary courses on electricity.

A Problem:

The generalized Coulomb law enables us to determine the electrostatic field $\vec{E}$ at point $\vec{r}$ provided we know the charge distribution $\rho(\vec{r}')$ that affects the

field at $\vec{r}$ over all space (or rather the relevant region of space close enough to $\vec{r}$ that its influence on $\vec{E}(\vec{r})$ is not negligible). However, in practice, most often we know $\rho(\vec{r}')$ only in a limited region of space beyond which we can spacify at most some boundary conditions on the fields (and not the exact form of the charge distribution) Clearly, in these cases the integral in the generalized Coulomb law cannot be evaluated explicitly:



$$\vec{E} = \int_V (\cdots) d^3\vec{r}' + \int_{\text{outside } V} (\cdots) d^3\vec{r}'$$

(unknown contribution)

Hence the formula we have is not useful in such situations. Before setting out to solve this problem, let us consider:

Some Properties of $\vec{E}$ as Given by the Generalized Coulomb Law:

① Using $\vec{\nabla} \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = 4\pi\delta(\vec{r}-\vec{r}')$ gives

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = k\int \rho(\vec{r}')\left(\vec{\nabla}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right) d^3\vec{r}' = k\int \rho(\vec{r}')\left(4\pi\,\delta(\vec{r}-\vec{r}')\right) d^3\vec{r}'$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi k\,\rho(\vec{r})}$$

(2) Since $\dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\left(\dfrac{1}{|\vec{r}-\vec{r}'|}\right)$,

$$\vec{E} = -\int \rho(\vec{r}')\,\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) d^3r' = -\nabla\left(\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'\right)$$

$$= -\nabla\phi \quad , \quad \text{where} \quad \phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

From here it follows that

$$\vec{\nabla}\times\vec{E} = -\nabla\times(\nabla\phi) \equiv 0$$

To summarize,

$$\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi k\rho, \qquad \vec{\nabla}\times\vec{E} = 0 \iff \vec{E} = -\nabla\phi}$$

Note that while $\vec{E}(\vec{r})$, as given by the generalized Coulomb law, depends on the form of $\rho(\vec{r}')$ at every space point, we have found two other properties of $\vec{E}$ i.e., $\vec{\nabla}\cdot\vec{E}(\vec{r})$ and $\vec{\nabla}\times\vec{E}(\vec{r})$ that at most depend on $\rho(\vec{r})$ at the same point $\vec{r}$ (and not other points). As a result, these equations have solutions that are more general than the "generalized coulomb law" form and resolve the problem mentioned above. However, since the above equations were based on the GCL, it is not clear that their new solutions should be trusted as describing real charge distributions and fields. Below

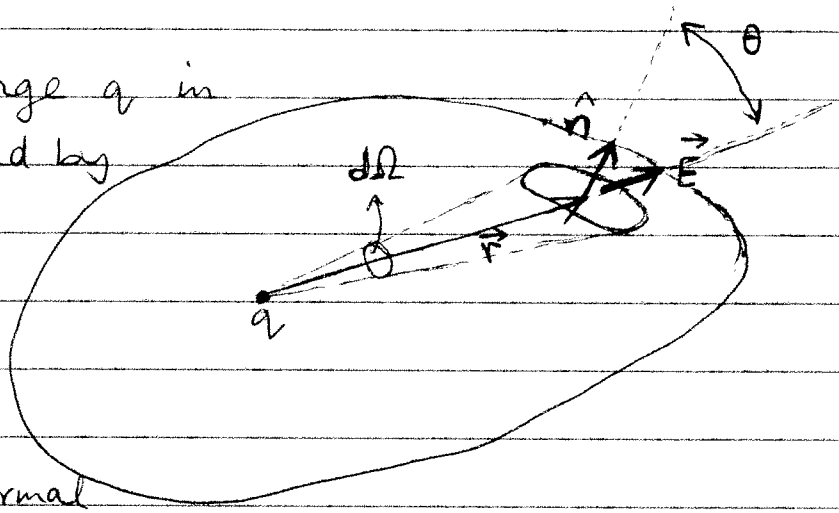
we show that these equations have a wider validity than GCL and that the new solutions are useful when the GCL form is not. We first discuss the $\vec{\nabla}.\vec{E}$ equation and then the $\vec{\nabla}\times\vec{E}$ eqn.

Gauss Law:

Suppose we know $\rho(\vec{r})$ only within a volume V. This is not enough to determine $\vec{E}$ which is also affected by charges outside V (of unknown distribution). The Gauss law provides us with a quantity which is only sensitive to $\rho(\vec{r})$ in V, or in the limit of small V, only to $\rho(\vec{r})$ at $\vec{r}$. These quantities are $\oint_S \vec{E}.d\vec{S}$ and $\vec{\nabla}.\vec{E}$.

Proof:

Consider a point charge q in a volume V bounded by a surface S. Let $\vec{r}$ be a vector from the charge q to the surface. Let $\hat{n}$ be the unit outward normal to S at the point $\vec{r}$. The electric field at $\vec{r}$ due to q is



$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{r^3}$$

Flux of $\vec{E}$ through surface element $d\vec{S} = \hat{n}\,dS$ is

$$\vec{E}.d\vec{S} = E.\hat{n}\,dS = kq\frac{\vec{r}.\hat{n}}{r^3}dS = k\frac{q}{r^2}\cos\theta\,dS$$

Now, consider a sphere of radius r centered at q. At the point $\vec{r}$ on the surface the unit normal to the sphere is $\hat{r}$ (the unit normal to surface S at the same point is $\hat{n}$). Then the projection of the area $\vec{dS}$ on the sphere is $\vec{dS} \cdot \hat{r} = dS \cos\theta$. This area element on the sphere subtends a solid angle at the position of the charge q given by

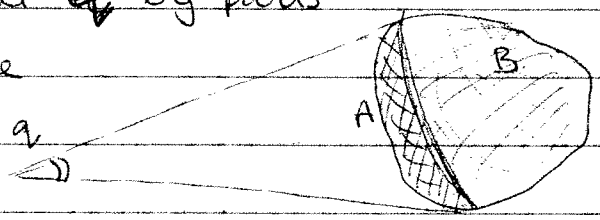
$$d\Omega = \frac{dS \cos\theta}{r^2}$$

Hence, $$\vec{E} \cdot \vec{dS} = \vec{E} \cdot \hat{n} \, dS = kq \, d\Omega$$

The total flux across S becomes

$$\oint_S \vec{E} \cdot \vec{dS} = kq \int d\Omega = 4\pi kq$$

For q outside S, $\int d\Omega = 0$. This is because the solid angles subtended at q by parts A and B of the surface are equal in magnitude but opposite in sign. So the net solid angle vanishes.



Then we have the Gauss Law

$$\oint_{S=\partial V} \vec{E} \cdot \vec{dS} = 4\pi kq \quad \text{for } q \text{ inside } V$$
$$= 0 \quad \text{for } q \text{ outside } V$$

In particular, charges outside V do not affect $\oint_S \vec{E} \cdot \vec{dS}$

For many discrete charges inside V,

$$\oint_S \vec{E}\cdot\vec{dS} = 4\pi k \sum_{i=1}^{n} q_i$$

and for a continuous charge distribution,

$$\boxed{\oint_S \vec{E}\cdot\vec{dS} = 4\pi k \int_V \rho(r)\, d^3r}$$ (irrespective of charge outside V)

The validity of the result depends on

1) the inverse square nature of Coulomb law
2) central nature of the force
3) the superposition property.

Differential form of Gauss Law:

Using divergence theorem,

$$\int_V (\vec{\nabla}\cdot\vec{E})\, d^3r = 4\pi k \int_V \rho(r) d^3r$$

For arbitrary V, this implies,

$$\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi k\, \rho(\vec{r})}$$

To derive this, it was enough to know $\rho(\vec{r})$ within V and not outside V.

$\vec{\nabla}\times\vec{E}$ and the Scalar Potential

Since $\vec{E}$ is defined as force per unit charge, $\int_A^B \vec{E}\cdot\vec{dl}$ computes the work done in moving a unit charge from

A to B along a given trajectory. If $\vec{E}$ is produced by a point charge, then the central nature of the force implies that this work depends only on the location of A and B and not on the trajectory connecting them. The superposition principle guarantees that the path independence of work also applies when $\vec{E}$ is produced by a number of point charges, or in the limit, by a continuous charge distribution (As we know continuous charge distributions arise as macroscopic descriptions of large numbers of point charge i.e. electrons and nuclei). In particular, if the charge is moved along a closed path, the work done is zero,

$$\oint_{\ell} \vec{E} \cdot d\vec{l} = 0$$

Then by Stokes's theorem, $\boxed{\vec{\nabla} \times \vec{E} = 0}$

From vector calculus we know that this also implies the existence of a scalar field ϕ such that

$$\boxed{\vec{\nabla} \times \vec{E} = 0 \iff \vec{E} = -\vec{\nabla}\phi}$$

The -ve sign here is a matter of convention.

The Meaning of ϕ :

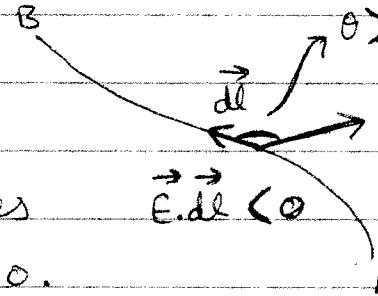
Let us now be more explicit about our conventions. The work done to move a point charge q from position

$\vec{r}_A$ to $\vec{r}_B$ along some curve l_{AB} in electric field $\vec{E}$ is

$$W_{AB} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot d\vec{l} = -q\int_{\vec{r}_A}^{\vec{r}_B} \vec{E}\cdot d\vec{l}$$

The sign is chosen such that work done <u>against</u> the field is positive. Hence W_{AB} represents the energy of the system (of $\vec{E}$ and charged particle) which increases if work is done on the system, say by moving q against $\vec{E}$, but it decreases when work is done by the system, say when $\vec{E}$ moves the charge.

For example, consider $q>0$. Remember that the direction of $\vec{E}$ is defined as one in which a +ve charge will move when placed in $\vec{E}$.



If we move the charge q against $\vec{E}$, then $\vec{E}\cdot d\vec{l} < 0$ and hence $dW = -q\vec{E}\cdot d\vec{l} > 0$. But if q moves along $\vec{E}$, then $\vec{E}\cdot d\vec{l} > 0$ and $dW < 0$. in this case $\vec{E}$ does the work and the energy of the system reduces. (For $q<0$, moving the charge against the field would correspond to $\vec{E}\cdot d\vec{l} > 0$ as $\vec{E}$ would tend to push the charge in the $-\vec{E}$ direction)

Now, $$W_{AB} = -q\int_{\vec{r}_A}^{\vec{r}_B} \vec{E}\cdot d\vec{l} = q\int_{\vec{r}_A}^{\vec{r}_B} \vec{\nabla}\phi\cdot d\vec{l} = q\int_{\vec{r}_A}^{\vec{r}_B} \frac{\partial\phi}{\partial l}\,dl$$

$$= q\int_{\vec{r}_A}^{\vec{r}_B} d\phi = q\phi(\vec{r}_B) - q\phi(\vec{r}_A)$$

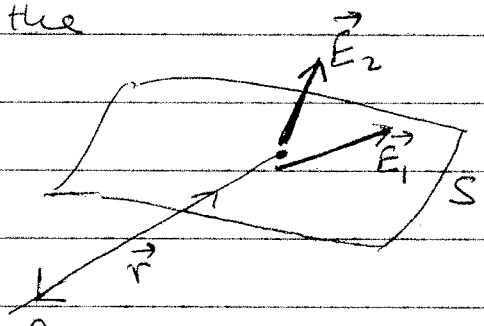
Hence $\phi(\vec{r})$ has the interpretation of potential energy per unit charge. It is called the scalar potential.

$\phi(\vec{r})$ is arbitrary upto an additive constant; $\phi(\vec{r})$ and $\phi(\vec{r}) + \text{Const}(c)$ give rise to the same $\vec{E}$ (since $\nabla c = 0$) and W_{AB} (where c cancels out).

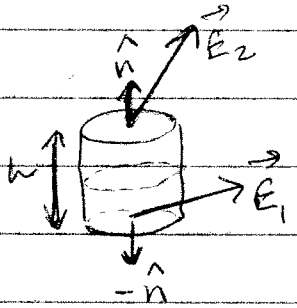
ϕ is additive for a number of charges.

An Application to Surface Charge Distribution

Consider a surface S of surface charge density $\sigma(\vec{r})$ (obviously, $\vec{r}$ is now a point on the surface S). $\vec{E}_2$ and $\vec{E}_1$ are the electric fields at points just above and just below $\vec{r}$, on two sides of the surface. To find the relation between $\vec{E}_2$ and $\vec{E}_1$, we construct a "pill box" through the surface as shown in the figure:



Applying Gauss law to the "pill box" gives



$$\oint \vec{E}\cdot d\vec{S} = \vec{E}_2\cdot\Delta\vec{S}_{top} + \vec{E}_1\cdot\Delta\vec{S}_{bottom} + \text{sides} = 4\pi\sigma\Delta S$$

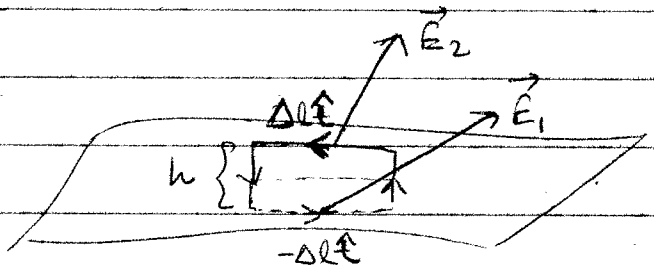
The contribution from the sides vanishes in limit $h \to 0$ and $\Delta\vec{S}_{top} = \Delta S\hat{n}$, $\Delta\vec{S}_{bottom} = -\Delta S\hat{n}$.

$$\therefore \quad \boxed{(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma}$$

Thus the surface charge density creates a discontinuity in the normal component of the electric field.

To see how the tangential component of $\vec{E}$ behaves, consider a small rectangular loop through the surface, (See figure) and evaluate

$$\oint \vec{E}\cdot d\vec{l} = \vec{E}_2\cdot\hat{t}\,\Delta l - \vec{E}_1\cdot\hat{t}\,\Delta l + \int_{(\text{sides } h)} \vec{E}\cdot d\vec{l} = 0$$



Again, in the limit $h \to 0$, the contribution from the sides of length h vanishes. Hence we have

$$\boxed{(\vec{E}_2 - \vec{E}_1)\,\hat{t} = 0}$$

Thus the tangential component of $\vec{E}$ remains continuous across the surface and is not affected by σ. Note that the orientation of the small loop is not fixed, so $\hat{t}$ could be any tangent to the surface.

Electrostatic Potential Energy of a Discrete Charge Distribution

The potential energy of a collection of n point charges q_i placed at positions $\vec{x}_i$ is the work done to assemble the collection by bringing in the charges from infinity to positions $\vec{x}_i$. It can be easily computed by assembling the collection one by one:

$n=2$: $W_2 = \dfrac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|}$ = potential energy of q_1 in the presence of q_2 or vice versa

$n=3$: $W_3 = \dfrac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \dfrac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \dfrac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|}$

$n=4$: $W_4 = \dfrac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} + \dfrac{q_1 q_3}{|\vec{x}_1 - \vec{x}_3|} + \dfrac{q_1 q_4}{|\vec{x}_1 - \vec{x}_4|}$

$+ \dfrac{q_2 q_3}{|\vec{x}_2 - \vec{x}_3|} + \dfrac{q_2 q_4}{|\vec{x}_2 - \vec{x}_4|}$

$+ \dfrac{q_3 q_4}{|\vec{x}_3 - \vec{x}_4|}$

$\vdots$

n : $W_n = \sum_{i=2}^{n} \dfrac{q_1 q_i}{|\vec{x}_1 - \vec{x}_i|} + \sum_{i=3}^{n} \dfrac{q_2 q_i}{|\vec{x}_2 - \vec{x}_i|} + \sum_{i=4}^{n} \dfrac{q_3 q_i}{|\vec{x}_3 - \vec{x}_i|} + \cdots$

$$\boxed{W_n = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}}$$

Electrostatic Potential Energy of a Continuous Charge Distribution:

The result for the discrete charge distribution cannot be directly generalized to the continuous case since the two are fundamentally different (as will be seen).

Let $\phi(\vec{x})$ be the potential due to a charge distribution $\rho(\vec{x})$. Now consider a very small change in the distribution, $\rho(\vec{x}) \to \rho(\vec{x}) + \delta\rho(\vec{x})$. The work done against $\phi(\vec{x})$ (which is assumed to remain unchanged in the process due to the smallness of $\delta\rho$) is

$$\delta W = \int_V \delta\rho(\vec{x})\,\phi(\vec{x})\,d^3x$$

$$= k \iint \frac{\delta\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x\,d^3x'$$

$$= \frac{1}{2}k \iint \frac{\delta\rho(\vec{x})\,\rho(\vec{x}') + \rho(\vec{x})\,\delta\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x\,d^3x'$$

$$= \frac{1}{2}k \iint \frac{\delta(\rho(\vec{x})\,\rho(\vec{x}'))}{|\vec{x}-\vec{x}'|}\,d^3x\,d^3x' \qquad \left(\text{use } \vec{x} \leftrightarrow \vec{x}'\right)$$

Then the total work done in building the charge distribution from $\rho = 0$ to some $\rho(\vec{x})$ is

$$\boxed{W[\rho] = \frac{1}{2}k \iint \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x\,d^3x'}$$

To verify this result, note that

$$W[\rho+\delta\rho] - W[\rho] = \delta W$$

where we have ignored terms of order $(\delta\rho)^2$ and higher.

Although the expression for $W[\rho]$ looks like the continuum generalization of the discrete case, the difference between the two becomes clear for $\rho(\vec{x}) = \sum_{i=1}^{n} q_i \, \delta(\vec{x}-\vec{x}_i)$. Then,

$$W = \frac{k}{2} \iint \frac{\sum_i q_i \delta(\vec{x}-\vec{x}_i) \sum_j q_j \delta(\vec{x}'-\vec{x}_j)}{|\vec{x}-\vec{x}'|} \, d^3x \, d^3x'$$

$$= \frac{k}{2} \sum_{i,j} \frac{q_i q_j}{|x_i - x_j|} \quad \text{(including the divergent } i=j \text{ terms)}$$

Hence the continuum formula is not valid for a discrete charge distribution as it leads to a divergent result. Naively, it may seem that the continuum expression also contains "self-energy" like effects because of the $\frac{1}{|\vec{x}-\vec{x}'|}$ factor in the integrand. However, as $|\vec{x}-\vec{x}'| \to 0$ the contribution of this factor to the integral is finite. To see this, define new variables

$$\vec{v} = \vec{x} - \vec{x}', \qquad u = \vec{x} + \vec{x}'$$

Then

$$W[\rho] \sim \iint \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|v|} |v|^2 \, dv \, d\Omega_v \, d^3u$$

where we have used $d^3v = |\vec{v}|^2 dv \sin\theta_v d\theta_v d\phi_v$ $\equiv |\vec{v}|^2 dv\, d\Omega_v$. Now, the integrand is clearly finite as $v \to \infty$ (in fact it vanishes in the limit) provided $\rho(\vec{x})$ and $\rho(\vec{x}')$ are finite in this limit. (the discrete limit was problematic since then as $v \to 0$, $\rho(x)\rho(x) \sim \delta(x-x_i)\delta(x-x_j)$ which for $x_i = x_j$ is sharply peaked and not smooth.)

The continuum result can also be written in other useful ways,

$$W = \frac{1}{2}\int \rho(\vec{x})\,\phi(\vec{x})\,d^3x \qquad \text{(note the } 1/2\text{)}$$

$$= \frac{1}{8\pi k}\int \vec{\nabla}\cdot\vec{E}(\vec{x})\,\phi(\vec{x})\,d^3x$$

$$= \frac{1}{8\pi k}\left(\int \vec{\nabla}\cdot(\vec{E}\phi)\,d^3x - \int \vec{E}\cdot\vec{\nabla}\phi\,d^3x\right)$$

$$= \frac{1}{8\pi k}\oint_{S=\partial V}(\vec{E}\phi)\cdot dS + \frac{1}{8\pi k}\int \vec{E}\cdot\vec{E}\,d^3x.$$

We can choose the integration region V large enough that the fields on its surface fall off to zero: Then the surface integral vanishes and we are left with

$$\boxed{W = \frac{1}{8\pi k}\int_V |\vec{E}|^2 d^3x}$$

From this, one can read off the <u>energy density</u> as

$$w = \frac{1}{8\pi k} |\vec{E}|^2$$

This expression is very general and also applies to regions of space where $\rho(\vec{x}) = 0$. It gives the energy stored in the electric field $\vec{E}$ due to the work done in setting up charge distributions that are the sources of $\vec{E}$.

Remark: We had two different expressions for W,

$$W = \frac{1}{2}\int \rho(x)\phi(x)\, d^3x = \frac{1}{8\pi k}\int |\vec{E}|^2 d^3x.$$

If we used the first expression to define an energy density, we would get $\boxed{w' = \frac{1}{2}\rho(x)\phi(x)}$ which vanishes for $\rho(x) = 0$ and hence is not consistent with w which need not vanish for $\rho(x) = 0$. In fact, w and w' differ by $\vec{\nabla}\cdot(\vec{E}\phi)$ which is the origin of the surface term in W. Thus, w and w' are equivalent from the point of view of computing the total energy of the system. However as energy densities, they are very different. The criterion that chooses w over w' is gauge invariance i.e. invariance under $\phi \to \phi + \text{const.}$ which should not affect physical quantities. Clearly, w' is affected by this transformation and cannot be physical, but w is invariant.

(End of Remark)

Note that W is always positive (this is not the case with U).

The form

$$W = \frac{1}{8\pi k} \int \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, d^3x$$

highlights the field theory connection of Maxwell's equations (in the static case)

Poisson and Laplace Equations

Let us get back to our basic equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi k\rho \,, \qquad \vec{\nabla} \times \vec{E} = 0$$

The second equation has the solution $\vec{E} = -\vec{\nabla}\phi$. In terms of ϕ the first equation becomes

$$\nabla^2\phi = -4\pi k\rho \qquad \text{(Poisson equation)}$$

$$= 0 \quad (\text{for } \rho = 0) \quad \text{(Laplace equation)}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(in cartesian coordinates)}$$

(*) Most problems in electrostatics reduce to solving these equations subject to appropriate boundary conditions. Large number of solutions exist.

(*) If $\rho(\vec{x})$ is known all over the space, then the solution is given by the generalized Coulomb law:

$$\phi(\vec{x}) = k \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

(*) If $\rho(\vec{x})$ is known only in a volume V, then the effects of charges outside V are encoded in the boundary conditions imposed on the solution on the boundary $S = \partial V$.

(*) General features of the solution and the role of boundary conditions are studied with the help of Green's identities and Green functions.

Green's Identities :

These are very useful in understanding the general features of the solutions of the Poisson and Laplace equations, especially the role of boundary conditions. Start with

$$\int_V (\vec{\nabla}.\vec{A})\, d^3r = \oint_S \vec{A}.\vec{ds} \qquad \text{(divergence theorem)}$$

Let

$$\vec{A} = \phi \vec{\nabla} \psi ,$$

then, $$\vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\vec{A}\cdot d\vec{s} = \phi(\vec{\nabla}\psi\cdot\hat{n})dS = \phi\frac{\partial\psi}{\partial n}ds$$

where $\hat{n}$ = unit normal to the surface element dS
and $\frac{\partial\psi}{\partial n}$ = directional derivative of ψ normal to dS.

Thus we have,

$$\int_V(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3r = \oint_S \phi\frac{\partial\psi}{\partial n}ds$$

(Green's first identity)

Interchanging ϕ and ψ and subtracting from the above equation, we get

$$\int_V(\phi\nabla^2\psi - \psi\nabla^2\phi)d^3r = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds$$

(Green's second identity)

Appearance of Boundary Conditions in the Solutions of $\nabla^2\Phi = -4\pi\rho$; A First Look

Start with Green's 2nd identity in a region of space spanned by the position vector $\vec{r}'$:

$$\int_V(\phi\nabla'^2\psi - \psi\nabla'^2\phi)d^3r' = \oint_S\left(\phi\frac{\partial\psi}{\partial n'} - \psi\frac{\partial\phi}{\partial n'}\right)ds'$$

choose: $\varphi = \Phi$ $\left(\nabla'^2 \Phi(\vec{r}') = -4\pi k \rho(\vec{r}')\right)$

and $\psi = \frac{1}{R} = \frac{1}{|\vec{r}-\vec{r}'|}$

where now: $\vec{r}$ = observation point

$\vec{r}'$ = integration variable in V.

Since,

$$\nabla'^2 \psi = \nabla_{r'}^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi \delta(\vec{r}-\vec{r}'),$$

we have,

$$\int_V \Phi\left(-4\pi \delta(\vec{r}-\vec{r}')\right) d^3r' + \int_V \frac{1}{|\vec{r}-\vec{r}'|} 4\pi k \rho(r')\, d^3r'$$

$$= \oint_S \left[\Phi \frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}_1|} \frac{\partial \Phi}{\partial n'}\right] ds'$$

or

$$\boxed{\Phi(\vec{r}) = k\int_V \frac{\rho(\vec{r}')}{\vec{r}-\vec{r}'} d^3r' + \oint_S \left(\frac{1}{4\pi}\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \frac{\Phi}{4\pi}\frac{\partial}{\partial n'}\frac{1}{R}\right) ds}$$

where $\frac{\partial \Phi}{\partial n'} = \nabla' \Phi \cdot \hat{n}'\Big|_S = -\vec{E}\cdot\hat{n}'\Big|_S$

$$\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) = \nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot \hat{n}' = \frac{\hat{R}\cdot\hat{n}}{R^2}$$

For $\Phi|_S = 0$, $\frac{\partial \Phi}{\partial n}\Big|_S = 0$ we recover the Coulomb law.

$\rho(r') = 0$: Laplace eqn.

To explicitly compute $\Phi(\vec{r})$ as given by the above expression, one has to specify both Φ and $\partial\Phi/\partial n$ on the boundary of V. But we will see below that these cannot be assigned independently and hence the above form of the solution is not practically useful.

In general solving second order differential equations may require specifying one of the following types of boundary conditions:

$\Phi|_S$, $\left.\frac{\partial\Phi}{\partial n}\right|_S$ (specified independently):

Cauchy b.c.'s.

$\Phi|_S$: Dirichlet b.c. ($\left.\frac{\partial\Phi}{\partial n}\right|_S$ is given by the sdn).

$\left.\frac{\partial\Phi}{\partial n}\right|_S$: Neumann b.c. ($\Phi|_S$ is determined by the sdn)

Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions

The above result shows how boundary conditions (the specification of Φ and $\partial\Phi/\partial n$ on the boundary) enter the solutions of Poisson or Laplace equations.

The question now is: How much boundary information does one need to uniquely specify

the solution (for a given $\rho(\vec{r}')$), but yet not over specify it ?

Now we show that supplying both $\Phi|_S$ and $\partial\Phi/\partial n|_S$ over specifies the soln. $\Rightarrow$ if the two are not consistent, then a soln does not exist. It suffices to supply either $\Phi|_S$ or $\frac{\partial\Phi}{\partial n}\Big|_S$. Thus the above form of the solution is not useful for practical purposes.

We now show the uniqueness of the solution to $\nabla^2\Phi = -4\pi k\rho$ in a volum V subject to Dirichlet or Neuman boundary conditions on $S = \partial V$. Let Φ_1 and Φ_2 be two solutions in V satisfying the same boundary conditions on S. Define

$$U = \Phi_2 - \Phi_1 \quad \Rightarrow \quad \nabla^2 U = 0 \quad (\text{in } V)$$

on U, the boundary conditions are

$$U|_S = 0 \quad (\text{Dirichlet}) \; ; \quad \frac{\partial U}{\partial n}\Big|_S = 0 \quad (\text{Neumann})$$

In Green's first identity, put $\Phi = \psi = U$:

$$\int_V \left(U\nabla^2 U + \vec{\nabla}U \cdot \vec{\nabla}U\right) d^3r = \oint_S U \frac{\partial U}{\partial n} dS$$

Thus for <u>either</u> boundary condition,

$$\int_V |\nabla U|^2 d^3r = 0 \quad \Rightarrow \quad \vec{\nabla}U = 0 \quad (\text{since } |\vec{\nabla}U|^2 \geq 0)$$

or $\boxed{U = \text{constant}}$ inside V.

For Dirichlet boundary conditions, $U|_S = 0 \Rightarrow U = 0$ and $\Phi_1 = \Phi_2$; QED.

For Neumann boundary conditions, $\frac{\partial U}{\partial n}\Big|_S = 0$ does not fix the value of the constant. Hence, $\Phi_2 = \Phi_1 + \text{Const}$, and the soln is unique upto an additive constant. But this is anyway the gauge ambiguity inherent in the definition of Φ. Hence in this case too, the solution is essentially uniquely specified.

The above proof also shows the uniqueness of the solution for mixed boundary conditions (i.e., Dirichlet over part of the surface and Neumann over the remaining part.

The uniqueness of the solution implies that no solutions exist for Cauchy boundary condition

Formal Soln of the Electrostatic Boundary-Value Problem with Green Functions :

We now have to write the soln in a form that involves only Dirichlet or Neumann b.c's (or appropriate mixed b.c's). This is achieved by using Green functions.

In the earlier attempt we used $\psi = \frac{1}{|\vec{r} - \vec{r}'|}$ in the Green's theorem, but all we really

needed was $\nabla^2 \psi = -4\pi\delta(\vec{r}-\vec{r}')$. So we could also have used a more general Green function $G(\vec{r},\vec{r}')$ defined by:

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

where, $$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

such that, $\nabla'^2 F(\vec{r},\vec{r}') = 0$ inside V

Now, in Green's theorem, we can set

$$\psi = G(\vec{r},\vec{r}')$$

The freedom due to the presence of $F(\vec{r},\vec{r}')$ can be used to eliminate either the $\phi|_S$ or the $\partial\Phi/\partial n|_S$ dependent term from the surface integral.

$G(\vec{r},\vec{r}')$ also solves the Poisson equation and hence requires boundary conditions specified on S. These should be simple and generic and should not depend on the detailed form of the b.c. on Φ.

Green's theorem with $\phi = \Phi$ and $\psi = G(\vec{r},\vec{r}')$ gives,

$$\Phi(\vec{r}) = \int_V \kappa\rho(\vec{r}')\, G(\vec{r},\vec{r}')\, d^3r' + \frac{1}{4\pi}\oint_S \left(G(\vec{r},\vec{r}')\frac{\partial\Phi}{\partial n'} - \Phi(r')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'} \right) ds$$

We now use the freedom of choosing F in G:

Dirichlet Case: For Dirichlet b.c. on Φ, demand,

$$\boxed{G_D(\vec{r},\vec{r}') = 0 \qquad \text{for } \vec{r}' \text{ on } S}$$

Then,

$$\boxed{\Phi(\vec{r}) = \kappa \int_V \rho(\vec{r}) G_D(\vec{r},\vec{r}')\, d^3r' - \frac{1}{4\pi} \oint_S \Phi(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, dS}$$

Neumann Case: Naively, one may set $\left.\frac{\partial G_N}{\partial n'}\right|_{\vec{r}' \in S} = 0$ (for $\vec{r}'$ on S) but Gauss law gives

$$\int_V \nabla'^2 G(\vec{r},\vec{r}')\, d^3r' = \oint_S \vec{\nabla}' G(\vec{r},\vec{r}') \cdot d\vec{S} = \boxed{\oint_S \frac{\partial G}{\partial n'}\, dS = -4\pi}$$

$\therefore$ the simplest allowed b.c. on G_N is

$$\boxed{\frac{\partial G_N(\vec{r},\vec{r}')}{\partial n'} = -\frac{4\pi}{S} \quad (\text{for } \vec{r}' \text{ on } S)}$$

where S in $4\pi/S$ is the area of the surface S.

Then,

$$\boxed{\Phi(\vec{r}) = \langle \Phi \rangle_S + \int_V \rho(\vec{r}') G_N(\vec{r},\vec{r}')\, d^3r' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} G_N\, ds'}$$

where $\langle \Phi \rangle_S = \frac{1}{S} \oint_S \Phi\, ds$: averge value of Φ over S.

Note that the boundary conditions on G_1 as simple and generic (independent of the b.c. for Φ). However G_1 does depend on the shape of the surface S in the problem.

Symmetry: G_D is symmetric, $G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r})$. G_N can be chosen to be symmetric as an additional requirement.

Physical Interpretation of $G_1(\vec{r}, \vec{r}')$:

The Green function itself can be regarded as an electrostatic potential: $\frac{1}{|\vec{r}-\vec{r}'|}$ is the potential at $\vec{r}'$ due to a unit charge placed at $\vec{r}$ within the volume V. In V, $\nabla^2 F = 0$. Hence $F(\vec{r}, \vec{r}')$ is a potential at $\vec{r}'$ due to a distribution of charges outside V. The outside distribution is chosen such that $G_1 = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r}, \vec{r}')$ satisfies the right boundary condition on S. This makes the outside charge distribution, and hence F, a function of $\vec{r}$ as well. This is the mathematical equivalent of the "method of images"

We will use the method of images to construct G_D for a simple problem.

The solutions of Laplace equation are also useful in constructing F.