Structure formation - Introduction

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Recap

• CMB photons provide a snapshot of the Universe at $z \sim 1090$ when it was 380 000 years old

• Temperature fluctuations are related to the potential of the density fluctuations as

\[
\left( \frac{\Delta T}{T} \right)_0 = \frac{1 + 3\alpha}{3 + 3\alpha} \phi = \frac{1}{3} \phi
\]

• We will investigate how perturbations grow until today where they manifest themselves as large scale filaments, galaxy clusters etc
Method

• We use perturbation theory to solve for small deviations from homogeneity

• Zeroth-order solution: FRW solution for homogeneous background
• First-order solution: Perturbations on top of the background

• We solve a model with only one species making up the cosmic fluid

• On scales smaller than $r_H = H^{-1}$, dynamics is essentially Newtonian, if the scale factor $a(t)$ is kept in the equations
Newtonian equations

• The continuity equation
  \[ \dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0 \]

• The Euler equation
  \[ \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla \phi - \frac{\nabla p}{\rho} \]

• The Poisson equation
  \[ \nabla^2 \phi = 4\pi \rho \]
$0^{th}$ order solutions

- $\rho_0(t, \bar{r}) = \rho_0 \left[ \frac{a_0}{a(t)} \right]^3$

- $\vec{v}_0(t, \bar{r}) = \frac{\dot{a}(t)}{a(t)} \bar{r}$

- $\phi_0(t, \bar{r}) = \frac{2\pi \rho_0 r^2}{3}$
When \( r \to \infty \)

- \( \rho_0, \nu_0 \) and \( \phi_0 \) all \( \to \infty \)
- \( \rho_0, \nu_0 \) and \( \phi_0 \) all stay finite
- \( \nu_0 \) and \( \phi_0 \) \( \to \infty \)
Structure formation – Fourier transforms

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Fourier transforms

• We can write functions as a sum of some set of basis functions
• In Taylor series, the basis functions are \((x - x_0)^n\)
• For Fourier transforms, the basis functions are sin or cos functions

\[ e^{i\omega t} = \cos \omega t + i \sin \omega t \]

\[ f(t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{i\omega t} f(\omega) d\omega \]

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1\textsuperscript{th} order equations

• We define a comoving space coordinate $\bar{x} \equiv \bar{r}[a_0/ a(t)]$

$$\rho(t, \bar{x}) = \rho_0(t) + \rho_1(t, \bar{x}) \equiv \rho_0(t)[1 + \delta(t, \bar{x})]$$

$$\bar{v}(t, \bar{x}) = \bar{v}_0(t) + \bar{v}_1(t, \bar{x})$$

$$\phi(t, \bar{x}) = \phi_0(t) + \phi_1(t, \bar{x})$$

• $\delta(t, \bar{x})$ is the density contrast
The density contrast is given by

\[ \delta \equiv \rho - \rho_0 \]

\[ \delta \equiv \frac{\rho - \rho_0}{\rho_0} \]

\[ \delta \equiv \frac{\rho}{\rho_0} \]
Fourier representation

- Describe the spatial behaviour of the density contrast $\delta(t, \bar{x})$
- Write as a sum of standing waves with time-dependent amplitude
- Fourier transform in terms of the wave vector $\vec{k}$ (where $k = \omega$)

$$\delta(t, \bar{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{i\vec{k} \cdot \bar{x}} \delta(t, \vec{k}) d^3k$$

with inverse

$$\delta(t, \vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-i\vec{k} \cdot \bar{x}} \delta(t, \bar{x}) d^3x$$

- Fourier components, $\delta(t, \vec{k})$, evolve independently
Fourier representation

- The gradient $\nabla$ corresponds to multiplication by $ik(a_0/a)$
- In Cartesian coordinates

$$\nabla^2 = \frac{d^2}{dr^2} = \left(\frac{a_0}{a}\right)^2 \frac{d^2}{dx^2} \rightarrow - \left(\frac{a_0}{a}\right)^2 k^2$$

- The first order equation in Fourier space is

$$\ddot{\delta}(t, k) + 2H \dot{\delta}(t, k) + \left[ k^2 v_s^2 \left(\frac{a_0}{a}\right)^2 - \frac{3H^2 \Omega_0}{2} \right] \delta(t, k) = 0$$

where the sound speed is

$$v_s^2 \equiv \left(\frac{\delta p}{\delta \rho}\right)$$
What is the sound speed of non-relativistic matter?

- $v_s \approx 0$
- $v_s = c/\sqrt{3}$
- $v_s \approx c$
Fouirer representation

• The Newtonian approach is valid for scales $r < r_H = H^{-1}$
• This translates to $k^2 \left( \frac{a_0}{a} \right)^2 \gg H^2$

• From the Poisson equation, $\delta(t, \bar{k}) = -\frac{2}{3} \left( \frac{k}{H} \frac{a_0}{a} \right)^2 \phi_1(t, \bar{k})$
The power spectrum

• The power spectrum is given by $P(k) = \langle \delta^2(k) \rangle$ where the average is over all directions $\bar{k}$

• Primordial fluctuations have $P(k) \propto k^n$ where $n \approx 1$

• The power spectrum is modified under the influence of gravity
Structure formation – The Jeans mass

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The role of pressure

- Perturbations on small scales are counteracted by pressure
- The density contrast is
  \[ \ddot{\delta}(t, \bar{k}) + 2H \dot{\delta}(t, \bar{k}) + \left[ k^2 v_s^2 \left( \frac{a_0}{a} \right)^2 - \frac{3H^2 \Omega_0}{2} \right] \delta(t, \bar{k}) = 0 \]

- The behaviour of \( \delta \) depends on \( \kappa_J \equiv \left[ k^2 v_s^2 \left( \frac{a_0}{a} \right)^2 - \frac{3H^2 \Omega_0}{2} \right] \)
  - If \( \kappa_J < 0 \), solutions will be growing
  - If \( \kappa_J > 0 \), solutions will be oscillating
The Jeans length and mass

• The wave number \( k \) correspond to a physical scale
  \[ \lambda = a/a_0 \cdot \frac{2\pi}{k} \]

• The Jeans length, \( \lambda_J \), is the physical scale for which \( \kappa_J = 0 \)

• The Jeans mass is the mass within a sphere of radius \( \lambda_J/2 \)

• Only perturbations larger than \( \lambda_J \) and \( M_J \) will grow with time
• Smaller scales are stabilised by pressure
The Jeans length and mass

- Before CMB decoupling, $v_s \approx c/\sqrt{3} \approx 0.58\ c$ and $\lambda_J \approx 0.6\ \text{Mpc}$
- The baryonic Jeans mass was $M_J \approx 7 \cdot 10^{18}\ M_\odot$

- Immediately after decoupling, the baryon gas has
  \[ v_s \approx \left(\frac{k_B T}{m_e}\right)^{1/2} c \approx 1.5 \cdot 10^{-5} c \]
  and $M_J \approx 10^5\ M_\odot$

- Perturbations in the baryon density smaller than supercluster scales, could not grow until the time of photon decoupling
What angular scale does \( \lambda_J \approx 0.6 \) Mpc at the time of CMB decoupling at \( z \approx 1090 \) correspond to?

- \( \theta_J \approx 3 \) arcsec
- \( \theta_J \approx 3 \) arcmin
- \( \theta_J \approx 3 \) deg
- \( \theta_J \approx 3 \) rad
Fully relativistic equations

• Valid on all scales, for pressureless perturbations ($\nu_s = 0$)
• The gravitational potential is given by
  \[ \ddot{\phi} + 4H\dot{\phi} + \left( H^2 + \frac{2\ddot{a}}{a} \right) \phi = 0 \]
• The density contrast is given by
  \[ \ddot{\delta} + 2H\dot{\delta} - 3(\ddot{\phi} + 2H\dot{\phi}) + \frac{k^2}{a^2} \phi = 0 \]
  and
  \[ \left[ \frac{1}{3} \left( \frac{k}{H} \frac{a_0}{a} \right)^2 + 1 \right] \phi + \frac{\dot{\phi}}{H} = -\frac{\Omega_M \delta}{2} \]
• In the Newtonian limit
  \[ \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2} H^2 \Omega_M \delta = 0 \]
In the Universe, density fluctuations are largest on

- small scales ($\ll 300$ Mpc)
- medium scales ($\sim 300$ Mpc)
- large scales ($\gg 300$ Mpc)
What density fluctuation does a spherical galaxy with $M \sim 10^{11} \, M_\odot$ and radius 10 kpc correspond to?

- $\delta \sim 10^{-6}$
- $\delta \sim 1$
- $\delta \sim 10^6$