New Class of Non-Abelian Spin-Singlet Quantum Hall States

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We present a new class of non-Abelian spin-singlet quantum Hall states, generalizing Halperin's Abelian spin-singlet states and the Read-Rezayi non-Abelian quantum Hall states for spin-polarized electrons. We label the states by (k, M) with M odd (even) for fermionic (bosonic) states, and find a filling fraction $\nu = 2k/(2kM + 3)$. The states with $M = 0$ are bosonic spin-singlet states characterized by a $SU(3)_k$ symmetry. We explain how an effective Landau-Ginzburg theory for the $SU(3)_2$ state can be constructed. In general, the quasiparticles over these new quantum Hall states carry spin, fractional charge and non-Abelian quantum statistics. [S0031-9007(99)09361-8]

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The developments that followed the discovery of the fractional quantum Hall effect in 1982 have challenged two traditional wisdoms on the nature of the quantum Hall states that are relevant for experimental observations. These two wisdoms concern the *spin* of the electrons that participate in a quantum Hall state and the *quantum statistics* of the quasiparticle excitations over these states.

Starting with the first wisdom, it is evident that under the conditions of the (fractional) quantum Hall effect, which happens in a strong magnetic field, there is an important Zeeman splitting between the energies of spin-up and spin-down polarized electrons. One may therefore expect that the observable quantum Hall states will be in terms of spin-polarized electrons. However, already in 1983, Halperin [1] pointed out that the energy associated with the Zeeman splitting is rather modest as compared to other energy scales in the system. Because of this, quantum Hall states which are not spin polarized but instead involve equal numbers of spin-up and spin-down electrons (forming a *spin singlet*) are feasible. The experimental confirmation of this idea came in 1989, when several groups reported that the ground states at $\nu = 4/3$ and $\nu = 8/5$ are spin unpolarized [2]. In recent experiments, which employed hydrostatic pressure to reduce the g factor, more detailed results on quantum Hall spin transitions were obtained [3]. In his 1983 paper [1], Halperin proposed the following spin-singlet (SS) quantum Hall states:

$$\tilde{\Psi}_{\rm SS}^{m+1,m+1,m}(z_1,\ldots,z_N;w_1,\ldots,w_N) = \Pi_{i<j}(z_i - z_j)^{m+1} \times \Pi_{i<j}(w_i - w_j)^{m+1}\Pi_{i,j}(z_i - w_j)^m, \quad (1)$$

where z_i and w_i are the coordinates of the spin-up and spin-down electrons, respectively. The state Eq. (1) has the filling fraction $\nu = 2/(2m + 1)$. Here and below we display reduced quantum Hall wave functions $\tilde{\Psi}(x)$, which are related to the actual wave functions $\Psi(x)$ via $\Psi(x) = \tilde{\Psi}(x)\exp(-\sum_i \frac{|x_i|^2}{4l^2})$ with $x_i = z_i, w_i$ and $l = \sqrt{\hbar c/eB}$ the magnetic length. It was emphasized in [4,5] that the wave function Eq. (1) can be factorized into a charge factor times a spin factor. The spin factor has an $SU(2)_1$ affine Kac-Moody symmetry and describes semionic spinons that are also encountered in other models of spin-charge separated electrons in $d = 1 + 1$ dimensions. More general (Abelian) spin-singlet states have been described in the literature [6].

Concerning the second wisdom, we remark that the traditional hierarchical quantum Hall states (Jain series) all share the property that the quantum statistics of their fundamental excitations are fractional but Abelian. While these states suffice to explain the overwhelming majority of experimental observations, there is the exception of the well-established quantum Hall state at $\nu = 5/2$, which does not fit into the hierarchical scheme. This observation has prompted the analysis of new quantum Hall states, the Haldane-Rezayi state [7] and the $q = 2$ pfaffian state [4] (both at $\nu = 1/2$) being the most prominent among them. The quasihole excitations over the pfaffian quantum Hall states satisfy what is called *non-Abelian statistics* [4,8]. By this, one means that the wave function describing a number of quasiholes at fixed positions has more than one component, and that the braiding of two quasiholes is represented by a matrix that acts on this multicomponent wave function. Since matrices in general do not commute, such statistics are called non-Abelian. The non-Abelian braid statistics of the quasiholes over the pfaffian quantum Hall states are reflected in their unusual exclusion statistics, i.e., in the appropriate generalization of the Pauli principle for particles of this type [9].

In a recent paper [10], Read and Rezayi proposed and studied a class of spin-polarized, non-Abelian quantum Hall states that generalize the pfaffian. Some of these states were independently considered by Wen [11]. The most general non-Abelian (NA) quantum Hall state studied by Read and Rezayi, with labels (k, M), is of the form

$$\tilde{\Psi}_{\rm NA}^{k,M}(z_1,\ldots,z_N) = \langle\psi(z_1)\cdots\psi(z_N)\rangle \times \Pi_{i<j}(z_i - z_j)^{(2/k)+M} \quad (2)$$

with $\psi(z)$ a so-called order-k parafermion and with the brackets $\langle\cdots\rangle$ denoting a correlator in the associated

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conformal field theory (CFT). The physical picture behind these wave functions is that of an instability involving a clustering of, at the most, k particles, generalizing the notion of "pairing" that underlies the pfaffian states [12].

In this Letter we describe a new class of quantum Hall states, which combine the feature of being a spin singlet with that of non-Abelian statistics. These new states can be viewed as non-Abelian generalizations of the spin-singlet states Eq. (1), or, alternatively, as spin-singlet analogs of the non-Abelian states Eq. (2). The wave function of our most general non-Abelian spin-singlet (NASS) state, labeled as (k, M) with $k > 1$, is displayed in Eq. (16) below. It has the filling fraction

$$\nu(k, M) = \frac{2k}{2kM + 3}. \tag{3}$$

The simplest fermionic NASS state (with $k = 2$, $M = 1$) occurs at filling fraction $\nu = 4/7$. In general, the NASS states are competing with Abelian spin-singlet states that are possible at the same filling fractions Eq. (3).

In recent studies of the pfaffian and Read-Rezayi non-Abelian quantum Hall states [13,14], it has been emphasized that the essential mechanism of their non-Abelian statistics is closely related to the presence of (a deformation of) a non-Abelian $SU(2)_k$ affine Kac-Moody symmetry with $k > 1$. In this Letter we shall see that, for the case of spin-singlet non-Abelian quantum Hall states, there is a very similar role for a symmetry $SU(3)_k$ with $k > 1$.

We start our presentation by a discussion of the non-Abelian $SU(3)_1$ symmetry of a particular Abelian spin-singlet quantum Hall state. In a recent paper [14], it was emphasized that the bosonic Laughlin state at $\nu = 1/2$ possesses a non-Abelian $SU(2)_1$ symmetry, which can be viewed as a continuous extension of the particle-hole symmetry at half-filling. We shall refer to this symmetry as SU(2) charge. In an earlier work by Balatsky and Fradkin [5], it was stressed that the $(1/2, 1/2, -1/2)$ Halperin state, which is a spin-singlet state at $\nu = \infty$, i.e., at $\mathbf{B} = 0$, possesses $SU(2)_1$ non-Abelian symmetry, which we here call SU(2) spin. Combining these two observations, one expects to find a bosonic spin-singlet quantum Hall state at *finite* ν, in which SU(2) charge and SU(2) spin combine into a nontrivial extended symmetry. The nature of this extended symmetry can be traced by analyzing the algebraic properties of creation and annihilation operators for spin-full hard-core bosons. We define

$$\begin{aligned} B_\sigma^\dagger &= \psi_{1\sigma}^\dagger \psi_2, \qquad B_\sigma = \psi_2^\dagger \psi_{1\sigma}, \\ B_a^3 &= \psi_{1\sigma}^\dagger \sigma_{\sigma\rho}^a \psi_{1\rho}, \qquad B^3 = \psi_{1\sigma}^\dagger \psi_{1\sigma} - 2\psi_2^\dagger \psi_2, \end{aligned} \tag{4}$$

with $\sigma, \rho = \uparrow, \downarrow$ and with $\psi_{1\sigma}^\dagger$, $\psi_{1\sigma}$ and $\psi_2^\dagger$, ψ_2 the creation and annihilation operators of spin-1/2 bosonic particles and holes. (We remark that in this setup the holes do *not* carry a spin index.) The "hard-core" condition is implemented by the constraint

$$\psi_{1\sigma}^\dagger \psi_{1\sigma} + \psi_2^\dagger \psi_2 = 1. \tag{5}$$

Using the defining commutators

$$[\psi_{1\sigma}, \psi_{1\rho}^\dagger] = \delta_{\sigma\rho}, \qquad [\psi_2, \psi_2^\dagger] = 1, \tag{6}$$

one shows that the eight operators B^A of Eq. (4) form an adjoint representation of the algebra SU(3). In standard mathematical notation, we denote by B_α the element of the Lie algebra SU(3) that corresponds to a root α. With simple roots $\alpha_1 = (\sqrt{2}, 0)$, $\alpha_2 = (-\sqrt{2}/2, \sqrt{6}/2)$, we identify the boson creation operators as

$$B_\uparrow^\dagger = B_{\alpha_1}, \qquad B_\downarrow^\dagger = B_{-\alpha_2}. \tag{7}$$

We conclude that the kinematics of hard-core spin-full bosons are organized by a SU(3) symmetry. In its fermionic incarnation SU(2|1), this symmetry is well known from the supersymmetric t-J model [15].

Alerted by this result, we quickly find that the $(2, 2, 1)$ bosonic spin-singlet state [Eq. (1) with $m = 1$] possesses a $SU(3)_1$ global symmetry. One way to see this is by recognizing that the inverse K-matrix, given by

$$K^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{8}$$

is equal to the inverse Cartan matrix of SU(3) up to a trivial change of sign. Working out the SU(3) structure of the edge CFT for the $(2, 2, 1)$ state, one identifies the Cartan subalgebra generators B^3 and B_3^3 with the spin and charge bosons according to

$$B^3 = i\sqrt{6}\,\partial\varphi_c, \qquad B_3^3 = -i\sqrt{2}\,\partial\varphi_s. \tag{9}$$

The fundamental quasiparticles reside in the triplet **3** and antitriplet $\bar{\mathbf{3}}$ representations of SU(3), with spin and charge quantum numbers

$$\begin{aligned} &\phi^1 : \text{spin } \uparrow, \qquad q = -1/3, \\ &\phi^2 : \text{spin } \downarrow, \qquad q = -1/3, \\ &\phi^3 : \text{spin } 0, \qquad q = 2/3, \end{aligned} \tag{10}$$

and opposite for the antitriplet. Following [16], we can construct the complete edge theory in terms of multiparticle states consisting of quanta of the fields ϕ^i, $i = 1, 2, 3$. The systematics of this construction lead to a notion of "fractional exclusion statistics" of these quasiparticles. In [17], the mathematical details of these fractional statistics, which differ from those proposed by Haldane [18], were presented. As a direct application of the results of [17], we may recover the Hall conductance σ_H of the $(2, 2, 1)$ state by working out the following expression [16]:

$$\sigma_H = n^{\max} q^2 \frac{e^2}{h}, \tag{11}$$

with $n^{\max}$ equal to the maximal occupation of a given single quasiparticle state. Substituting the values $q = 2/3$, $n^{\max} = 3/2$ for the positive-charge carriers ϕ^3, we recover the value $\sigma_H = \frac{2}{3}\frac{e^2}{h}$.

An effective Landau-Ginzburg bulk theory for the $(2,2,1)$ state can be cast in the following form:

$$\mathcal{L} = |DB|^2 + V(B) + \mathcal{L}_{\mathrm{CS}}(a) + \epsilon^{\mu\nu\lambda} A^{\mathrm{ext}}_{\mu} f^3_{\nu\lambda}, \tag{12}$$

where B is the SU(3) octet Bose field, DB is the covariant derivative in the adjoint representation, $V(B)$ is a potential, and a^A_μ is a SU(3) Chern-Simons (CS) gauge field. The external field A^{ext}_μ couples to the f^3 component of the field tensor of the gauge field a^A_μ. The Chern-Simons Lagrangian is given by

$$\mathcal{L}_{\mathrm{CS}}(a) = \frac{1}{4\pi}\,\epsilon^{\mu\nu\lambda}\Big(a^A_\mu \partial_\nu a^A_\lambda + \frac{2}{3} f_{ABC} a^A_\mu a^B_\nu a^C_\lambda\Big). \tag{13}$$

For the justification of the result Eq. (12) we refer to [14], where an analogous result for the $\mathrm{SU}(2)_1$ invariant $\nu = 1/2$ state was presented.

Having understood the $\mathrm{SU}(3)_1$ structure of a particular Abelian SS state, we can proceed with the construction of NASS states. Closely following the logic presented in [14] (see also [19]), we first consider a state with symmetry $\mathrm{SU}(3)_2$, which we obtain by performing a (dual) coset reduction on two copies of the $\mathrm{SU}(3)_1$ state. As explained in [14], this reduction procedure leads to a statistical transmutation of one of the octet fields and renders the statistics of the triplet quasiparticles ϕ^i non-Abelian.

The effective edge CFT for the $\mathrm{SU}(3)_2$ theory is completely determined by the known structure of the $\mathrm{SU}(3)_2$ chiral Wess-Zumino-Witten (WZW) theory. At the same time, this CFT can be used to generate an explicit expression for the ground state wave function in the bulk [see Eq. (16) below], and for the wave functions representing various quasiparticle excitations. The spin and charge quantum numbers of the fundamental triplet excitations are the same as those listed in Eq. (10), but the exclusion statistics are now non-Abelian.

Before presenting more general NASS states, we remark that an effective Landau-Ginzburg theory for the $\mathrm{SU}(3)_2$ NASS state is readily given, by generalizing the construction of [14] for the bosonic pfaffian state with symmetry $\mathrm{SU}(2)_2$. The Landau-Ginzburg theory is obtained by taking two copies of the $\mathrm{SU}(3)_1$ theory Eq. (12) and using a pairing mechanism that is similar to the electron pairing in a Bardeen-Cooper-Schrieffer theory. The pairing induces a symmetry breaking $\mathrm{SU}(3)_1 \times \mathrm{SU}(3)_1 \to \mathrm{SU}(3)_2$ and hence induces a level $k > 1$ non-Abelian symmetry that is characteristic of non-Abelian statistics. One may check that the stable vortices in the broken-symmetry theory correspond to the quasiparticles that are identified using the edge CFT.

We now turn to the description of a two-parameter family of NASS states. They are obtained by taking a backbone $\mathrm{SU}(3)_k$ theory, dressed with an additional Laughlin factor with exponent M. To obtain explicit expressions for the corresponding wave functions, we rely on the SU(3) parafermions introduced by Gepner in [20]. These parafermions, written as ψ_α, are labeled by *roots* α of SU(3) and have the property that $\psi_\alpha = \psi_\beta$ when α-β is an element of k times the root lattice. In terms of the ψ_α and of two auxiliary bosons $\varphi_{1,2}$, the affine currents $B_\alpha(z)$ at level k are written as

$$B_\alpha(z) \propto \psi_\alpha \exp(i\alpha\varphi/\sqrt{k})(z). \tag{14}$$

Using the identification Eq. (7), we arrive at the following expression for the NASS state associated with $\mathrm{SU}(3)_k$:

$$\begin{aligned}\lim_{z_\infty\to\infty} (z_\infty)^{6(N^2)/k} \langle \exp[iN/\sqrt{k}\,(\alpha_2 - \alpha_1)\varphi](z_\infty) \\ \times B_{\alpha_1}(z_1)\cdots B_{\alpha_1}(z_N) B_{-\alpha_2}(w_1) \\ \cdots B_{-\alpha_2}(w_N)\rangle. \end{aligned} \tag{15}$$

Substituting the form Eq. (14), one observes that the correlator factorizes as a parafermion correlator times a factor coming from the vertex operators. The latter is seen to combine into the $(1/k)$th power of the $(2,2,1)$ spin-singlet wave function. Multiplying the result with an overall Laughlin factor, we arrive at the following wave function for the (k,M) NASS state:

$$\begin{aligned}\tilde{\Psi}^{k,M}_{\mathrm{NASS}}(z_1,\ldots,z_N;w_1,\ldots,w_N) \\ = \langle \psi_{\alpha_1}(z_1)\cdots\psi_{\alpha_1}(z_N)\psi_{-\alpha_2}(w_1)\cdots\psi_{-\alpha_2}(w_N)\rangle \\ \times [\tilde{\Psi}^{2,2,1}_{\mathrm{SS}}(z_i;w_j)]^{1/k}\tilde{\Psi}^{M}_{\mathrm{L}}(z_i;w_j), \end{aligned} \tag{16}$$

with $\tilde{\Psi}^{2,2,1}_{\mathrm{SS}}$ as in Eq. (1) and $\tilde{\Psi}^{M}_{\mathrm{L}}$ the standard Laughlin wave function with exponent M, with odd (even) M giving a fermionic (bosonic) state. By combining the final two factors in Eq. (16), one recognizes a two-layer state with label $(M + 2/k, M + 2/k, M + 1/k)$, and since it is these factors that determine the filling fraction ν we immediately derive the value given in Eq. (3).

For $k = 1$, the state Eq. (16) reduces to the Abelian SS state Eq. (1) with $m = M + 1$. For $k > 1$, the parafermion correlator is nonvanishing for N, an integer multiple of k. The simplest nontrivial example of a wave function of the type Eq. (16) is the case $(k = 2, M = 0)$ for a total of $N = 4$ bosonic particles. We find

$$\begin{aligned}\tilde{\Psi}^{k=2,M=0}_{\mathrm{NASS}}(z_1,z_2;w_1,w_2) = 2(z_1z_2 + w_1w_2) \\ - (z_1 + z_2)(w_1 + w_2). \end{aligned} \tag{17}$$

By inspecting the zeros of this wave function, we can understand the pairing that underlies this particular quantum Hall state (compare with [4,10,12]). We see that upon sending $z_2 \to z_1$ or $w_1 \to z_1$ the wave function Eq. (17) does *not* go to zero. However, as soon as three or more particles come together, we do get a zero. [For three particles of the same spin, i.e., three z_i or three w_i, this cannot be seen from Eq. (17); it follows, however, from the operator

product expansion structure of the SU(3) parafermions ψ_α.] We conclude that the pairing of the $k = 2$ NASS states is similar to that of the pfaffian states for spin-polarized electrons. By the analogy with the findings of [10], one similarly expects that the instability underlying the level-k NASS states for $k > 2$ will be a "k-particle clustering."

The CFT underlying the states Eq. (16) is unitary, and the bulk-edge correspondence for these states thus avoids some of the subtleties that arise for the Haldane-Rezayi NASS state. It is therefore straightforward to derive the spin and charge quantum numbers of the fundamental quasiparticles over these states, and to determine exponents for quasiparticle and electron edge tunneling processes. For $M \neq 0$ the $SU(3)_k$ symmetry is broken and SU(3) quantum numbers are no longer meaningful. The fundamental flux-$\frac{1}{2k}$ quasiholes carry charge $q = \pm 1/(2kM + 3)$ and spin-$1/2$. Their conformal dimension, which is obtained by adding contributions from the parafermion sector and from the spin and charge sectors, equals

$$\Delta_{\rm qh} = \frac{(5k - 1)M + 8}{2(k + 3)(2kM + 3)}. \qquad (18)$$

The edge electrons have charge -1, spin-$1/2$, and conformal dimension $\Delta_{\rm el} = (M + 2)/2$, independent of k. The non-Abelian braid and exclusion statistics of the various quasiparticles follow by a straightforward generalization of the techniques of [4,8,9,17].

We remark that some of the filling factors for which fermionic NASS states exist agree with values for which spin transitions have been seen in experiments. With regard to the competition between Abelian and non-Abelian spin-singlet quantum Hall states with the same filling fraction ν, we remark the following. Based on explicit numerical work, the authors of [10] have suggested that, in the second Landau level, non-Abelian spin-polarized states tend to be favored over their Abelian counterparts. By analogy, one may expect that the NASS states proposed in this Letter will be favored over Abelian SS states when $\nu > 2$. On the basis of the reasoning presented in [12], one also expects that the $k = 2$ NASS states will be particularly relevant for samples with wide well or double well geometries. We finally remark that, experimentally, one can in principle distinguish between Abelian and non-Abelian quantum Hall states by studying processes where a current tunnels through the quantum Hall medium. Such experiments probe conformal dimensions such as Eq. (18), and these in general differ between Abelian and non-Abelian states with the same filling fraction.

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