Entanglement entropy from CFT:

Entanglement: divide system in two parts A, B $\to \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Assume system is in a pure state $|\Psi\rangle$. A and B are entangled if $|\Psi\rangle \neq |\psi\rangle_A \otimes |\phi\rangle_B$

Simplest example. $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right)$

Entanglement entropy: Von Neumann: $S_A = -\mathrm{Tr}\, \rho_A \log \rho_A$, with $\rho_A = \mathrm{Tr}_B\, \rho$

Perform Schmidt decomposition: $|\Psi\rangle = \sum_i \alpha_i\, |\psi_i\rangle_A \otimes |\phi_i\rangle_B$ (note the single sum!).

α_i: non-neg. real numbers, $\sum_i \alpha_i^2 = 1$ (normalisation).

$V = A, B$: $\rho_V = \sum_i \alpha_i^2 |\psi_i\rangle_V \langle \psi_i|$

$$S_V = S_A = S_B = -\mathrm{Tr}\, \rho_V \log \rho_V = -\sum_i \alpha_i^2 \log(\alpha_i^2)$$

The scaling of S_A from CFT: A is interval of length ℓ: $(L \to \infty)$

$$S_A = \frac{c}{3} \log \frac{\ell}{\varepsilon} + c_1'$$

c = central charge
ℓ : length of interval
ε : cutoff scale
c_1' : non-universal number.

In PBC:

$$S_A = \frac{c}{3} \log\left(\frac{L}{\pi \varepsilon} \sin\left(\frac{\pi \ell}{L}\right)\right) + c_1'$$

In d-dimensional system: $S_A \sim \ell^{d-1}$. In 1-d, at critical point: universal log term!

Strategy: (See Cardy & Calabrese: J. Stat. Mech P06002 (2004)
J. Phys. A 42, 504005 (2009) (review).)

First, introduce the Rényi entropies:

$$S_A^{(n)} = \frac{1}{1-n} \ln \operatorname{Tr} \rho_A^n ; \qquad \lim_{n\to 1} S_A^{(n)} = S_A$$

λ_i eigenvalues of ρ_A, $\sum \lambda_i = 1$, so $\operatorname{Tr} \rho_A^n = \sum_i \lambda_i^n$ converges for $\operatorname{Re} n > 1$
and analytic

$$S_A = - \lim_{n\to 1} \frac{\partial}{\partial n} \operatorname{Tr} \rho_A^n$$

How to calculate ρ_A^n for real n? Calculate for integer n, and use existence & uniqueness of analytic continuation.

For $\operatorname{Tr} \rho_A^n$ n integer: use replica trick!

↳ partition function on complicated Riemann surface (of simple Lagrangian).

or: part. function on $\mathbb{C}$ of complicated, but tractable problem.

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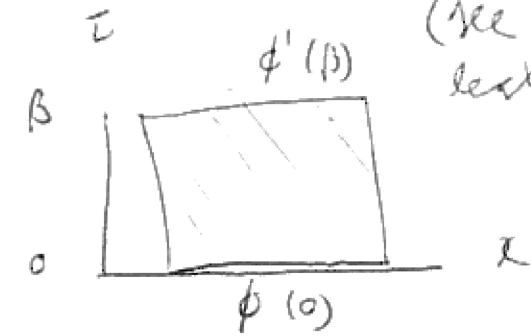
(see previous lecture).

The calculation is done via path integrals:

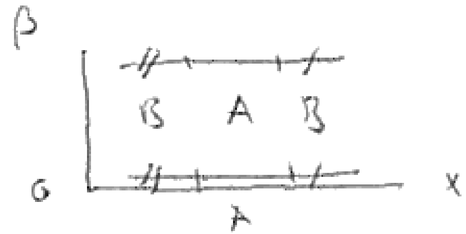


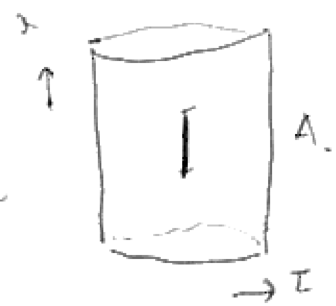
$$\rho(\phi,\phi') = \frac{1}{Z}\int_{BC.} [D\phi]\; e^{-\int_0^\beta d\tau\, L[\phi]}$$

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

$Z =$ 

$\rho_A = \mathrm{Tr}_B\, \rho$: obtained by sewing the open boundaries along B.

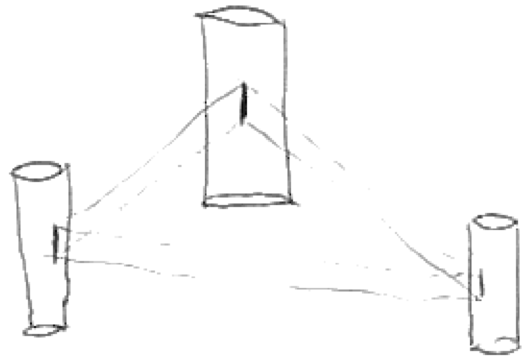
$\rho_A = \dfrac{1}{Z}$ 

, or 

$\mathrm{Tr}\, \rho_A^n$: take n copies, sew the cuts together cyclicly: ($1 \le j \le n$)

$$\phi_j(x, \tau=\beta^-) = \phi_{j+1}(x, \tau=0^+) \quad \text{for } x \in A.$$
$$\phi_n(x, \tau=\beta^-) = \phi_1(x, \tau=0^+)$$

$R_n =$



$Z_n(A)$ is part. function on R_n.

$$\mathrm{Tr}\, \rho_A^n = \frac{Z_n(A)}{Z^n}$$

R_n is flat, except for branch points,

Strategy: transform cal. of Z on R^n to calculate Z on $\mathbb{C}$, with correct BC's around the branch points (nonzero curvature).

$$Z_R = \int [d\phi]\, e^{-\int_R dx\, d\tau\, \mathcal{L}[\phi](x,\tau)}$$

$$= \int_{C_A} [d\phi_1] \cdots [d\phi_n]\, e^{-\int_{\mathbb{C}} dx\, d\tau \left(\mathcal{L}[\phi_1] + \cdots + \mathcal{L}[\phi_n] \right)}$$

$\cancel{\mathcal{L}^{(n)} = \mathcal{L}}$

$\mathcal{L}^n[\phi_1, \ldots \phi_n] = \mathcal{L}[\phi_1] + \ldots + \mathcal{L}[\phi_n]$

C_A: $\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-)$ $\quad x \in A = [u, v]$

The fields ϕ_i are twist fields, $\mathcal{L}^{(n)}$ invariant under cyclic permutations.

To take care of these BC's, we can insert fields at the branchpoints:

$$\Phi_n \equiv \Phi_\sigma \quad : \quad \sigma : i \to i+1 \mod n$$
$$\Phi_{-n} \equiv \Phi_{\sigma^{-1}} \quad \sigma : i \to i-1 \mod n.$$

For partition function.

$$\frac{Z_{R_n}}{Z^n} \propto \langle \Phi_n(u,0)\, \Phi_{-n}(v,0) \rangle_{\mathcal{L}^{(n)},\mathbb{C}}$$

General correlation functions: $\langle O(x,\tau,\text{sheet } i) \cdots \rangle_{R^n} = \dfrac{\langle \Phi_n(u,0)\, \Phi_{-n}(v,0)\; O_i(x,\tau) \cdots \rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \Phi_n(u,0)\, \Phi_{-n}(v,0) \rangle_{\mathcal{L}^{(n)},\mathbb{C}}}$

We take O_i to be T_i, and use the Ward identity to calculate Z_{R_n}.

On R_n: $w = x + i\tau$, $\bar{w} = x - i\tau$ $\qquad w \to \zeta = \dfrac{(w-u)}{(w-v)}$

Fold open by $\zeta \to z = \zeta^{1/n} = \left(\dfrac{w-u}{w-v}\right)^{1/n}$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\} \quad , \quad \{z,w\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

$\langle T(z)\rangle_{\mathbb{C}} = 0$, so we find:

$$\langle T(w)\rangle_{R_n} = \frac{c}{24}\,\frac{\overset{(1-1/n^2)}{\cancel{(n^2-1)}}\,(u-v)^2}{(u-w)^2(v-w)^2} \Rightarrow$$

This also equals $\langle T(w)\rangle_R = \dfrac{\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\,T(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}$

The stress-energy tensor of $\mathcal{L}^{(n)}$ get a factor of n:

$$\frac{\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\,T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}} = \frac{c(n^2-1)}{24n}\,\frac{(u-v)}{(w-u)^2(w-v)^2}$$

The conformal ward identity reads:

$$\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\,T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = \left(\frac{h_{\Phi_n}}{(w-u)^2} + \frac{\partial_u}{w-u} + \frac{h_{\Phi_{-n}}}{(w-v)^2} + \frac{\partial_v}{(w-v)}\right)\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}},$$

w/ $h_{\Phi_{\pm n}} = d_n$ the scaling dimension of $\Phi_{\pm n}$.

non-chiral!

One obtains: $\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = (u-v)^{-4d_n}$, with $d_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$

$\operatorname{Tr}\rho_A^n \propto \frac{Z_n(A)}{Z^n}$, so $\operatorname{Tr}\rho_A^n = c_n\left(\frac{v-u}{\epsilon}\right)^{-\frac{c}{6}(n-1/n)}$

ϵ inserted for dimensional reasons, renormalises Z

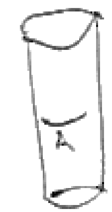
c_n cannot be fixed in general, but $\operatorname{Tr}\rho_A = 1$ gives $c_1 = 1$

$$S_A^{(n)} = \frac{1}{1-n}\log \mathrm{Tr}\,\rho_A^n \qquad l = u-v$$

$$= \frac{c}{6}\left(\frac{n+1}{n}\right)\log\frac{l}{\epsilon} + c_n' \quad , \text{ and } S_A = \frac{c}{3}\log\frac{l}{\epsilon} + c_1'$$

$$c_n' = \frac{\log c_n}{1-n}$$

Because $\langle \Phi_n(u,0)\,\Phi_{-n}(v,0)\rangle_C$ is a correlator of primaries, we can map it to cylinder. $w \to z = \frac{L}{2\pi}\log w$, and take a spacial cut:



$$\mathrm{Tr}\,\rho_A^n = c_n\left(\frac{L}{\pi\epsilon}\sin\left(\frac{\pi l}{L}\right)\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \Rightarrow S_A = \frac{c}{3}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi l}{L}\right)$$