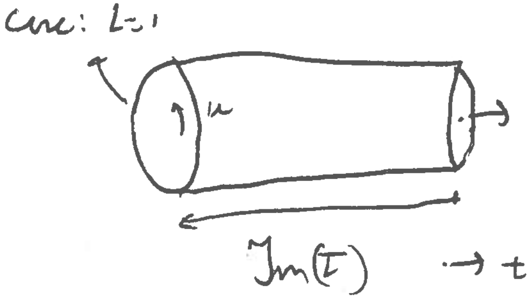
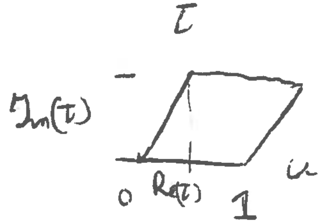
Constructing a torus from a cylinder; with parameter τ:

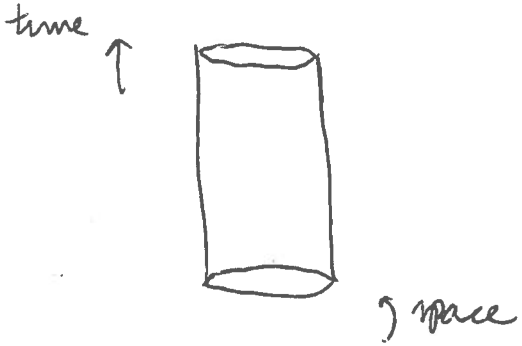


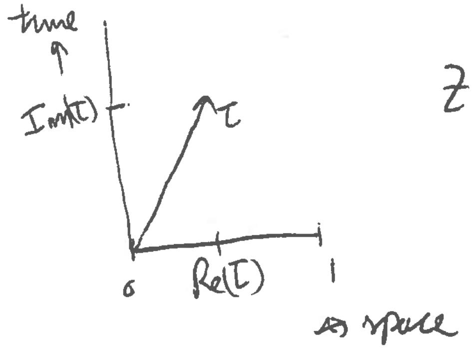


Twist RHS of cylinder by $\mathrm{Re}(\tau)$, and glue.

Hamiltonian (gen. of t-translations) on cylinder: $H = \frac{2\pi}{L}(L_0 + \bar{L}_0) - \frac{\pi c}{6L}$

Momentum (" " u- ") $P = \frac{2\pi}{L}(L_0 - \bar{L}_0)$





The partition function:

$$Z = \underset{\text{PBC}}{\mathrm{Tr}}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\mathrm{Im}(\tau) H}$$

" $t=\beta$ "

$\mathrm{Re}(\tau) = 0$

$\mathrm{Re}(\tau) \neq 0 \Rightarrow$ Rotation over $\mathrm{Re}(\tau)$ generated by $e^{iP\,\mathrm{Re}(\tau)}$

$$\Rightarrow Z = \mathrm{Tr}\, e^{-\mathrm{Im}(\tau) H + i\,\mathrm{Re}(\tau) P}$$

To get the partition function, we note that on a τ-torus, a time translation is accompanied by a translation over $\mathrm{Re}(\tau)$, so we get:

$$Z = \mathrm{Tr}\, e^{-\mathrm{Im}(\tau)\, H + i\, \mathrm{Re}(\tau)\, P}$$

$$= \mathrm{Tr}\, e^{\pi \frac{c}{6} \tau_2}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)}\, e^{-2\pi \tau_2 (L_0 + \bar{L}_0)} = e^{\pi \frac{c}{6} \tau_2}\, \mathrm{Tr}\left(e^{2\pi i \tau L_0}\, e^{-2\pi i \bar{\tau} \bar{L}_0}\right)$$

$\tau = \tau_1 + i\tau_2$

$$= (q\bar{q})^{-c/24}\, \mathrm{Tr}\, q^{L_0}\, \bar{q}^{\bar{L}_0}, \quad \text{where } q = e^{2\pi i \tau}$$

Trace is over all scaling fields in the ~~theo~~ theory, labeled by $(h_i + N, \bar{h}_i + \bar{N})$

conf. family

descendant.

For minimal models, we have:

$$Z = \sum_{r,s;\bar{r},\bar{s}} n_{r,s;\bar{r},\bar{s}}\, \chi_{r,s}(q)\, \chi_{\bar{r},\bar{s}}(\bar{q}) \stackrel{?}{=} \sum n_{r,s;\bar{r}\bar{s}}\, \chi_{r,s}(\hat{q})\, \chi_{\bar{r},\bar{s}}(\hat{\bar{q}})$$

For which n's is this true?

χ's transform as: $\chi_{r,s}(\hat{q}) = \sum_{r',s'} S_{r,s}^{r',s'}\, \chi_{r',s'}(q)$, with

$$S_{rs}^{r's'} = \left(\frac{8}{pp'}\right)^{\frac{1}{2}} (-1)^{1+rs'+s'r} \sin\left(\frac{\pi p r r'}{p'}\right) \sin\left(\frac{\pi p' s s'}{p}\right)$$

S: real, sym, w/ $S^2 = 1$

Proof: use Poisson resummation

One solution ~~that~~ for n's that always exists: $n_{r,s;\bar{r}\bar{s}} = \delta_{r,\bar{r}}\, \delta_{s,\bar{s}}$

But: there are other solutions!!

The partition function can be written as:

$$Z = \sum_{\substack{h, \bar{h} \\ \text{families}}} n_{h,\bar{h}} \, \chi_h(q) \, \chi_{\bar{h}}(\bar{q})$$

$n_{h,\bar{h}}$: number of primaries with $(h, \bar{h})$ as its lowest weight

$$\chi_h(q) = q^{-c/24 + h} \sum_{N=0}^{\infty} d_h(N) \, q^N$$

$d_h(N)$: deg. at level N

Invariance of Z under T is simple: requires that $h - \bar{h}$ is integer.

Invariance of Z under S is non-trivial: Z is the same powerseries in $q, \bar{q}$, as in $\hat{q} = e^{-2\pi i/\tau}$ and $\bar{\hat{q}}$. (Very) non-trivial!

S-matrix $\leftrightarrow$ fusion rules!!

Poisson ressummation (strong $\leftrightarrow$ weak coupling duality):

Integrate $\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x}$ over $e^{-\pi a x^2 + bx}$,

resulting in: $\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{-\pi/a\left(k + \frac{b}{2\pi i}\right)^2}$

CFT & critical chains:

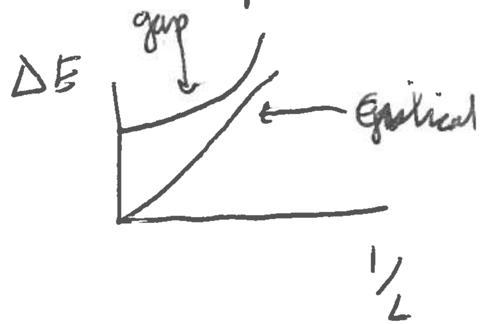
Example: $H = -J \sum_i \left(g\, \sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right)$

From the exact solution, we know that the system is critical for $g=1$

In general, one needs to determine first if the system is critical or not ...

* Determine the low-lying part of the spectrum for various system sizes

Gap behaves as :



↳ $\Delta E = E_{1,\text{ex}} - E_{gs}$.

; $E_{gs} = \underbrace{E_0 L}_{\text{extensive contribution}} - \frac{\pi v c}{6L}$

v → velocity

c → central charge

Scaling of g.s. energy gives E_0 & vc

General form of E_i : $E_i = E_0 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + h_L + h_R\right)$

$h_L = h + n$

$h_R = \bar{h} + \bar{n}$

$h, \bar{h}$: scaling dim of primaries

$n, \bar{n}$: non-neg. integers

* One can determine v and h from a low-lying primary and 1^{st} descendant; knowing v also gives c.
* determine h for other primaries

→ Extracting the scaling behaviour of the energies is typically hard.

→ The momenta of the primaries are <u>not</u> fixed by conformal invariance.

Having the spectrum for large enough L allows one to check if a certain CFT fits the spectrum.

* low lying states ⟷ low lying contributions to the partition function

$$Z = \sum_{\substack{h,\bar{h} \\ \text{families}}} n_{h,\bar{h}}\, \chi_h(q)\, \chi_{\bar{h}}(\bar{q}) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}}$$

$$\chi_h(q) = q^{h - \frac{c}{24}} \sum_{n=0}^{\infty} \underbrace{d_h(n)}_{\text{degeneracy}}\, q^n$$

Ising: $\chi_{0,0} = 1 + q^2 + q^3 \Rightarrow \chi_{0,0}(q)\,\chi_{0,0}(\bar{q}) = 1 + q^2 + (\bar{q})^2 + q^3 + \bar{q}^3 + \cdots$

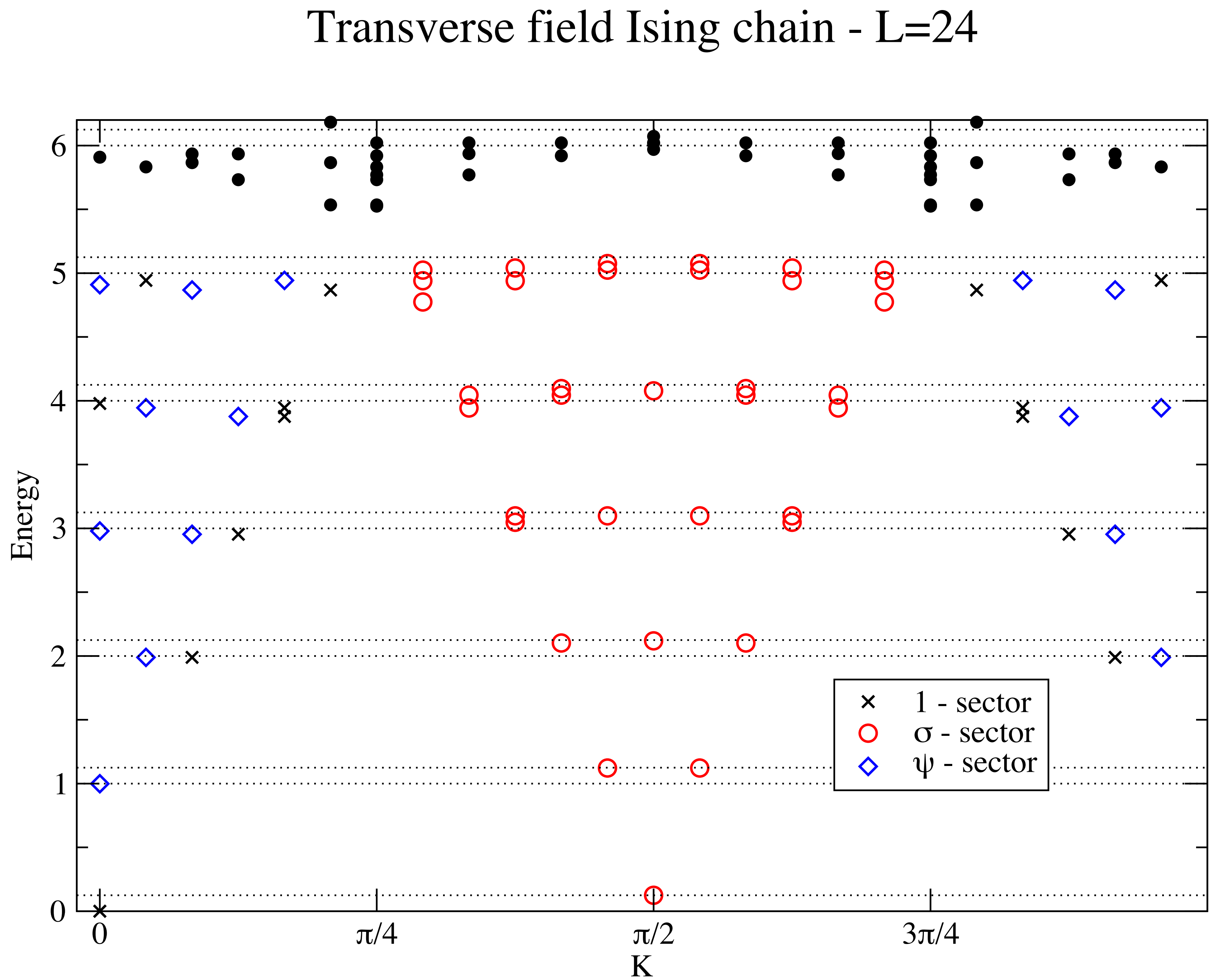
($-\frac{c}{24}$: shifted away)

$$Z_{\frac{1}{16},\frac{1}{16}} \Rightarrow q^{\frac{1}{16}}\, \bar{q}^{\frac{1}{16}} \left(1 + \underbrace{q + \bar{q}} + \underbrace{q\bar{q} + q^2 + \bar{q}^2} + \cdots + 2q^3 + q^2\bar{q} + q\bar{q}^2 + 2\bar{q}^3 + \cdots\right)$$

$$Z_{\frac{1}{2},\frac{1}{2}} \to q^{\frac{1}{2}}\, \bar{q}^{\frac{1}{2}} \left(1 + q + \bar{q} + q^2 + q\bar{q} + \bar{q}^2 + q^3 + q^2\bar{q} + q\bar{q}^2 + \bar{q}^3 + \cdots\right)$$

<u>Total</u> power of q & $\bar{q}$ gives the energy of each state.

<u>Difference</u> of the powers of q & $\bar{q}$ gives the relative momentum to the momentum of the primary field


Transverse field Ising chain - L=24
Energy
K
0
1
2
3
4
5
6
0
π/4
π/2
3π/4
1 - sector
σ - sector
ψ - sector