More general class of minimal models $\mathcal{M}(p,p')$

* finite number of primaries
* ~~not~~ no unitarity constraint

$$c(p,p') = 1 - \frac{6(p'-p)^2}{pp'}$$ p, p': co-prime $\quad |p-p'|=1$: unitary case

$$h_{r,s}(p,p') = \frac{(rp-sp')^2-(p'-p)^2}{4pp'}$$ $r = 1, \dots, p'-1$, $s = 1, \dots, p-1$

Again $\exists$ two fold redundancy in the labels: $\phi_{r,s} \equiv \phi_{p'-r,p-s}$

The fields $\phi_{r,s} \cong \phi_{p'-r,p-s}$ have a null descendant at levels rs <u>**and**</u> $(p'-r)(p-s$

Null vectors gave constraints ~~of~~ on 3 point correlators:

$C_3 = \langle \phi_{2,1}(z_1)\, \phi_{r,s}(z_2)\, \phi_{r',s'}(z_3) \rangle$ is non zero <u>only if</u>

$(r', s') = (r \pm 1, s)$

$C_3' = \langle \phi_{1,2}(z_1)\, \phi_{r,s}(z_2)\, \phi_{(r',s')}(z_3) \rangle$ non zero <u>only if</u> $(r', s') = (r, s \pm 1)$

Conditions are not sufficient; we write:

$$\phi_{(2,1)} \times \phi_{(r,s)} = \phi_{(r-1,s)} + \phi_{(r+1,s)}$$

$$\phi_{(1,2)} \times \phi_{(r,s)} = \phi(r, s-1) + \phi_{(r,s+1)}$$

In particular: $\cancel{\phi_{1,2}}$

$$\phi_{(2,1)} \times \phi_{(1,2)} = \phi_{(0,2)} + \phi_{(3,2)}$$

$$\phi_{(1,2)} \times \phi_{(2,1)} = \phi_{(2,0)} + \phi_{(2,2)}$$

Associativity gives $\phi_{(1,2)} \times \phi_{(2,1)} = \phi_{(2,2)}$. The ~~wd~~ 'would be' fields $\phi_{(0,2)}$ and $\phi_{(2,0)}$ are absent!

Fusing repeatedly with $\phi_{(1,2)}$ and $\phi_{(2,1)}$ leads to the following 'fusion rules':

$$\phi_{r,s} \times \phi_{r',s'} = \sum_{r''=|r-r'|+1}^{\min(r+r'+1,\, 2p'-1-r-r')} \; \sum_{s''=|s-s'|+1}^{\min(s+s'+1,\, 2p-1-s-s')} \phi_{r'',s''}$$

Product of two conformal families decomposes into a finite 'sum' of conformal families.

Consistency of the null vectors gives the truncation of the fields !

Minimal models: rational CFT-s, # primaries is finite.

Minimal model $\mathcal{M}(3,4)$: critical point of 2D Ising (classical)

(1D quantum TFI).

2D Ising correlation functions:

$$\langle \sigma_n \sigma_0 \rangle \sim \frac{1}{|n|^{d-2+\eta}} = \frac{1}{|n|^{\eta}} \quad \text{spin-spin correlator}$$

$\mathcal{M}(3,4)$: $\qquad h$:

$1 = \phi_{1,1} = \phi_{3,2} \quad (0,0$

$\sigma = \phi_{2,1} = \phi_{2,2} \quad (\frac{1}{16}, \frac{1}{16}$

$\psi = \phi_{1,2} = \phi_{3,1} \quad (\frac{1}{2}, \frac{1}{2}$

Energy: $\langle \varepsilon_n \varepsilon_0 \rangle \simeq \langle \sigma_n \sigma_{n+1} \sigma_0 \sigma_1 \rangle \sim \dfrac{1}{|n|^{2(d-1/\nu)}}$

Result: $\eta = \frac{1}{4}$; $\nu = 1$ $\Rightarrow$ $\langle \sigma_n \sigma_0 \rangle \sim \dfrac{1}{|n|^{1/4}} = \dfrac{1}{r^{2\Delta_\sigma}}$ $\qquad \Delta_\sigma = h_\sigma + \bar{h}_\sigma = 2h_\sigma$

$$\langle \varepsilon_n \varepsilon_0 \rangle \sim \frac{1}{|n|^{2\Delta_\varepsilon}} \sim \frac{1}{r^{2\Delta_\varepsilon}} \qquad \Delta_\varepsilon = h_\psi + \bar{h}_\psi = 2h_\psi$$

Other correlators can be expressed in terms of η and ν

Fusion rules of Ising CFT: $\quad 1 \times x = x \qquad x \in \{1, \sigma, \psi\}$

$(a \times b = b \times a)$ $\qquad \sigma \times \sigma = 1 + \psi$

$$\sigma \times \psi = \sigma$$

$$\psi \times \psi = 1$$

The correlator of a bunch of ψ's can be calculated using Wick's theorem (ψ is a free field; fermionic):

Using $\langle \psi \rangle = 0$, and $\langle \psi(z_1)\psi(z_2)\rangle = \dfrac{1}{(z_1 - z_2)}$, one obtains:

$$\langle \psi(z_1)\psi(z_2)\cdots\psi(z_N)\rangle = \mathcal{A}\left[\langle \psi(z_1)\psi(z_2)\rangle \cdots \langle \psi(z_{N-1})\psi(z_N)\rangle\right] \quad \left(\begin{matrix}\text{Assume:}\\ N \text{ even}\end{matrix}\right)$$

$$= \mathcal{A}\left[\frac{1}{(z_1 - z_2)}\frac{1}{(z_3 - z_4)}\cdots\right]$$

$$:= \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \propto \sqrt{\det\left(\frac{1}{z_i - z_j}\right)}$$

Obtaining four point correlator w/ σ fields: $\sigma = \phi_{2,1}$

Consider the chiral part:

$$\mathcal{G}(\{w\}) = \langle \phi_{2,1}(w_1)\, \phi_{2,1}(w_2)\, \phi_{2,1}(w_3)\, \phi_{2,1}(w_4) \rangle$$

The null state at level two gives, for $i=1,..,4$ and with $h = \frac{1}{16}$

$$\left[\frac{3}{2(2h+1)} \partial^2_{w_i} + \left(\sum_{j \neq i} \frac{\partial_{w_j}}{(w_j - w_i)} - \frac{h}{(w_i - w_j)^2} \right) \right] \mathcal{G}(\{w\}) = 0$$

Using conf. symmetry: $\mathcal{G}(\{w\}) = (w_1 - w_2)^{-2h} (w_3 - w_4)^{-2h} \mathcal{F}(x)$,

one can write

where $x = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{w_{12} w_{34}}{w_{14} w_{32}}$; $1 - x = \frac{w_{13} w_{42}}{w_{14} w_{32}}$

The coupled PDE's reduce to an ODE !

Two ways of showing this: 1) Take the appropriate lin. comb. of the 4 PDE's

2) Set $w_1 = 0$; $w_2 = x$; $w_3 = \infty$; $w_4 = 1$

In the Ising case, we write: $\langle \sigma(w_1) \dots \sigma(w_4) \rangle = w_{12}^{-\frac{1}{8}} \, w_{34}^{-\frac{1}{8}} \, (1-x)^{-\frac{1}{8}} \, H(x)$

$H(x)$ satisfies:

$$\left[x(1-x)\partial_x^2 + \left(\tfrac{1}{2} - x\right)\partial_x + \tfrac{1}{16} \right] H(x) = 0$$: Hypergeometric equation

Similar equation ~~s~~ can be found for 4-point correlator of $\phi_{1,2}$ and $\phi_{2,1}$ for arbitrary minimal model.

In this case, a ~~...~~ change of variable ~~...~~ leads to:

$$H^{\pm}(x) = \frac{1}{\sqrt{2}} \sqrt{\pm 1 + \sqrt{1-x}} \, , \quad \text{so} \quad \tilde{f}^{\pm}(x) = \frac{1}{\sqrt{2}} (1-x)^{-\frac{1}{8}} \sqrt{\pm 1 + \sqrt{1-x}}$$

What do the two solutions correspond to?

Correlators: expectation values w.r.t. to vacuum, The overall fusion channel should be the identity! (Otherwise $\mathcal{G} = 0$).

This can be satisfied in two ways. For instance, combining (1,2) and (3,4):

$$\sigma(w_1) \times \sigma(w_2) = 1 \quad \& \quad \sigma(w_3) \times \sigma(w_4) = 1$$

or

$$\sigma(w_1) \times \sigma(w_3) = \psi \quad \& \quad \sigma(w_3) \times \sigma(w_4) = \psi$$

The OPE:

$$\lim_{w_1 \to w_2} \sigma(w_1)\sigma(w_2) = (w_1 - w_2)^{-\frac{1}{8}} \mathbb{1} + (w_1 - w_2)^{3/8} C_{\sigma\sigma\psi}\, \psi(w_2) + \text{h.o.t.}$$

Note $C_{\sigma\sigma 1} = 1$, normalization of $\sigma(w)$.

Consider the limits: $w_1 \to w_2$ and $w_3 \to w_4$. Then: $x \to 0$; $1-x \to 1$

One finds: $\lim\limits_{\substack{w_1 \to w_2 \\ w_3 \to w_4}} w_{12}^{1/8} w_{34}^{1/8} g^+ = 1$

Consistent with $\sigma(w_1) \times \sigma(w_2) = 1$

$\sigma(w_3) \times \sigma(w_4) = 1$

g^+: ~~\&~~ 1 fusion channel of $\sigma(w_1)$ & $\sigma(w_2)$

$\langle \psi(w_2)\, \psi(w_4) \rangle =$

$$\lim_{\substack{w_1 \to w_2 \\ w_3 \to w_4}} w_{12}^{-3/8} w_{34}^{-3/8} g^- C_{\sigma\sigma\psi}^{-2} = \lim w_{12}^{-\frac{1}{2}} w_{34}^{-\frac{1}{2}} \frac{(1-x)^{-\frac{1}{8}}}{\sqrt{2}} \underbrace{\sqrt{-1+\sqrt{1-x}}}$$

Takin the lim gives $\dfrac{C_{\sigma\sigma\psi}^{-2}}{2} \dfrac{1}{(w_2 - w_4)} = \langle \psi(w_2)\psi(w_4) \rangle$ $\to \frac{1}{\sqrt{2}}\sqrt{-x} = \frac{1}{\sqrt{2}}\sqrt{-\frac{w_{12} w_{34}}{w_{14} w_{32}}}$

$= \frac{1}{\sqrt{2}}\sqrt{\frac{w_{12} w_{34}}{w_{14} w_{23}}}$

So we find $C_{\sigma\sigma\psi} = \frac{1}{\sqrt{2}}$, up to a sign.