Last week:

States vs. operators: $|\phi_j\rangle = \phi_j(0)|0\rangle$, with $\phi_j(z)$ a primary field.

Descendant states: $L_{-n_k} \cdots L_{-n_1}|\phi_j\rangle$

The L_n: modes of $T(z)$: $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$

The corresponding ~~op~~ descendant operators: $\hat{L}_{-n}\phi_j(w) = \oint \frac{dz}{2\pi i} \frac{1}{(z-w)^{n-1}} T(z)\phi_j(w)$

because, w/ ~~the~~ this def: $\hat{L}_{-n}\phi(0) = \oint \frac{dz}{2\pi i} z^{-n+1} T(z)\phi_j(0)$, such that

$$\hat{L}_{-n}\phi(0)|0\rangle = L_{-n}\phi_j(0)$$

$$= L_{-n}|\phi_j\rangle, \text{ so}$$

$\hat{L}_{-n}\phi(w)$ creates the descendant state $L_{-n}|\phi_j\rangle$

A primary field φ_i, and all its descendants $\hat{L}_{-n_k} \cdots \hat{L}_{-n_1} \varphi_i$ form a 'conformal family' $[\varphi_i]$.

The full OPE reads:
$$\varphi_i(z,\bar{z})\,\varphi_j(w,\bar{w}) = \sum_{\substack{p \\ \text{families}}} \sum_{\{k,\bar{k}\}} C_{ijp}^{\{k,\bar{k}\}}\, z^{h_p - h_i - h_j + \sum_l k_l}\, \bar{z}^{\bar{h}_p - \bar{h}_i - \bar{h}_j + \sum_l \bar{k}_l}\, \varphi_p^{\{k,\bar{k}\}}(w,\bar{w})$$

The ope coefficients $C_{ijp}^{\{k,\bar{k}\}}$ can be obtained from C_{ijp}; $\{h_i\}_i$ (no descendants) by conf. invariance

The C_{ijp}'s (no descendants) give the 3-point functions:
$$\langle \varphi_i | \varphi_j(z,\bar{z}) | \varphi_p \rangle = \langle \varphi_i(\infty)\, \varphi_j(z,\bar{z})\, \varphi_p(0) \rangle = C_{ijp}\, z^{-(h_p - h_i - h_j)}\, \bar{z}^{-(\bar{h}_p - \bar{h}_i - \bar{h}_j)}$$

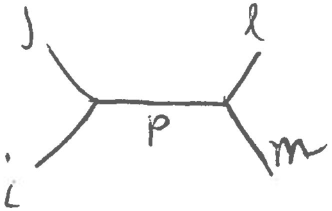
One needs four point functions to calculate the C_{ijp}'s.

Rational CFT: finite number of primaries. Necessary condition for being able to specify the CFT in full!!

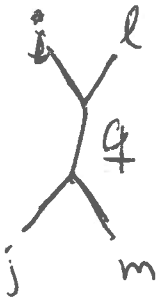
Conformal bootstrap: try to find other consistency conditions and solve them to obtain CFT's:

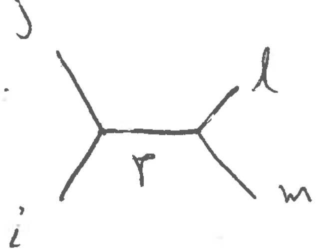
global constraint on 4-point function:

$$\langle \phi_i(z_1)\,\phi_j(z_2)\,\phi_l(z_3)\,\phi_m(z_4)\rangle$$: two ways to evaluate:

1: set $z_2 \to z_1$; $z_4 \to z_3$ Schematically (use OPE twice) : $\sum_p$  $C_{ijp}\, C_{lmp}$

$\Big)\Big)$ # of eq^{ns}.

2: $z_3 \to z_1$; $z_4 \to z_2$ $\sum_q$  $C_{ilq}\, C_{jmq}$



$= \mathcal{F}^{jl}_{im}(p, x)$

conformal block $x = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

Bootstrap: relates $\mathcal{F}$'s at x with $\mathcal{F}$'s at $1-x$.

Representations of the Virasoro algebra.

Unitary ireps of $su(2)$: * Take highest weight state, act w/ lowering operators
* Non-negative norm: integer & half integer spins.

Non-negative norm gives constraints on c, h:

$$[L_2, L_{-2}] = 4L_0 + \frac{c}{12}\cdot 2\cdot 3 \Rightarrow \frac{c}{2} = \langle 0|[L_2, L_{-2}]|0\rangle$$

$$L_{-2} = L_2^+ \qquad = \langle 0| L_2 L_2^+ |0\rangle = \| L_2^+|0\rangle\|^2 \geq 0.$$

So, we find that for unitary reps, we have $c \geq 0$

Norm of $L_{-n}|h\rangle$: $\langle h|L_n L_{-n}|h\rangle = \left[2nh + \frac{c}{12}n(n^2-1)\right]\langle h|h\rangle$
$n \geq 0$

n large: $c > 0$ take $n=1$; $h=0$: $L_{-1}|h\rangle = 0$, so $|h\rangle$ w/ $h=0$ is the vacuum

$n=1$: $h \geq 0$

Strategy: construct unitary reps. of Virasoro at some value's of c,

Find: possible at discrete set of c's; * finite # of irreps
* Not all states $L_{-n_k} \cdots L_{-n_1}|h\rangle$ are independent: null states
↳ give rise to diff eq^ns for correlators of primaries.

<u>Null states:</u>

At level one: $L_{-1}|h\rangle = 0$ (demand) $\Rightarrow h = 0$

At level two: $|\chi\rangle = L_{-2}|h\rangle + \alpha (L_{-1})^2|h\rangle = 0$, for some α.

Consider:
$$
\begin{aligned}
L_1|\chi\rangle &= [L_1, L_{-2}]|h\rangle + \alpha\,[L_1,(L_{-1})^2]|h\rangle \\
&= 3L_{-1}|h\rangle + \alpha\,(2L_0L_{-1} + L_{-1}L_1L_{-1})|h\rangle \\
&= 3L_{-1}|h\rangle + \alpha\,(2L_0L_{-1} + 2L_{-1}L_0 + (L_{-1})^2\cancel{L_1})|h\rangle \\
&= [3 + 2\alpha(2h+1)]\,L_{-1}|h\rangle = 0 \Rightarrow \alpha = \frac{-3}{2(2h+1)}
\end{aligned}
$$

$$L_2|\chi\rangle = [L_2, L_{-2}]|h\rangle + \alpha\, [L_2, (L_{-1})^2]|h\rangle$$

$$= \left(4L_0 + \frac{c}{12}\,6\right)|h\rangle + 3\alpha\, L_1 L_{-1}|h\rangle$$

$$= \left(4h + \frac{c}{2} + 6\alpha h\right)|h\rangle = 0 \;\Rightarrow\; c = -2\,(4+6\alpha)\,h = \frac{2h(5-8h)}{(2h+1)}$$

So, for $c = \frac{2h(5-8h)}{(2h+1)}$, the highest weight state $|h\rangle$ satisfies

$$\left(L_{-2} - \frac{3}{2(2h+1)}(L_{-1})^2\right)|h\rangle = 0$$

We say that $|h\rangle$ has a null descendant at level 2, or is 'degenerate' at level two

The associated field satisfies $\underbrace{\left(\hat{L}_{-2} - \frac{3}{2(2h+1)}(\hat{L}_{-1})^2\right)\phi_h}_{\chi(z)} = 0$

The correlator of the field ϕ_h is annihilated by $\left[\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\left(\mathcal{L}_{-1}\right)^2\right]$

One can show: $|\chi\rangle$ is orthogonal to the whole Verma module of $|h\rangle$, spanned by $L_{-n_k} \cdots L_{-n_1}|h\rangle$.

Let $X = \phi_1(z_1) \cdots \phi_n(z_n)$, then we have $\langle \chi(z) X \rangle = 0$, or

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\left(\mathcal{L}_{-1}\right)^2\right) \langle \phi_h(z) X \rangle = 0.$$

Using $\mathcal{L}_{-1} = \frac{\partial}{\partial z}$, we find

$$\left[-\frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{n} \left(\frac{h_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i}\right)\right] \langle \phi_h(z) X(z_i) \rangle = 0$$

Null descendants give rise to diff. eqn^s for the primary fields!