Exercises CFT-course fall 2023, set 10.

1. CFT description of the transverse-field Ising model.

   In the first exercise sheet, we considered the transverse-field Ising model in one dimension, with periodic boundary conditions. The hamiltonian in terms of the Pauli matrices σ was given by
$$H = -\sum_i \left(h\sigma_i^z + J\sigma_i^x\sigma_{i+1}^x\right) .$$

   You showed that the hamiltonian can be diagonalized, i.e., written as
$$H = \sum_k \varepsilon_k \left(2\gamma_k^\dagger\gamma_k - 1\right) . \tag{1}$$

   The model is critical for $|J| = |h|$, but here we set $h = J = 1$. Then we have $\varepsilon_k = (2 - 2\cos(2\pi k/L))^{\frac{1}{2}}$. The $\gamma_k^{(\dagger)}$'s are fermion creation and annihilation operators with momentum k. For F odd, the momenta take the values $k = 0, 1, \ldots, L-1$, while for F even, the allowed momenta are $k = 1/2, 3/2, \ldots, L - 1/2$, because of the periodic boundary conditions we imposed.

   Using the explicit solution of the model, it is, with some effort, possible to obtain the energies of the low-lying states analytically. It is often the case, however, that one only knows these energies numerically (for instance, using DMRG). In this exercise, you can choose if you want to do the analysis analytically, or numerically, by using the exact solution to numerically evaluate the energies of the low-lying states for system sizes up to, say $L = 500$. In the text below, it is assumed you do the exercise 'the numerical way'. It should be obvious what you have to do in case you do the exercise 'the analytical way'. Note that generically, it might be quite a bit harder to determine from numerical data whether a one-dimensional system is critical or not. If the system is indeed critical, it can still be hard to determine which CFT describes the system.

   a. Evaluate the the ground state energy (for $h = J = 1$) numerically for finite system sizes L and fit the results to the (general) finite size scaling formula
$$E_i = E_0 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + h_l + h_r + n\right) ,$$
   where E_0 is a constant energy per spin, v a velocity, c the central charge, and h_l and h_r the left and right scaling dimensions of the scaling fields. Finally, n is a non-negative integer. For the primary fields, $n = 0$, while for the descendants, $n \geq 1$.

   b. Show numerically that the system is critical, by obtaining the finite size gap between the ground and first excited states.

   c. Obtain the other constants in the scaling formula and the dimensions of the primary fields by considering various low-lying states in the spectrum.

2. Entanglement entropy from conformal field theory.

Study the paper 'Entanglement entropy and conformal field theory' by P. Calabrese and J. Cardy (J.Phys. A**42**, 504005 (2009); arXiv:0905.4013) up to (at least) section 3.2 (the article is available on the course website under 'reading material').

Briefly describe the structure of the calculation, and fill in the details which were skipped in the (old) lecture notes posted online (lecture-14.pdf), in particular the calculation of $\langle T(w)\rangle_{\mathcal{R}_n}$ and the correlator $\langle \Phi_n(u,0)\Phi_{-n}(v,0)\rangle_{\mathcal{L}^{(n)},\mathbf{C}}$.