Exercises CFT-course fall 2023, set 8.

1. In this exercise, we will derive an ordinary differential equation for the four point function of $\phi_{2,1}$ of the minimal models:

$$G(\{w_i\}) = \langle \phi_{2,1}(w_1)\phi_{2,1}(w_2)\phi_{2,1}(w_3)\phi_{2,1}(w_4)\rangle = (w_1-w_2)^{-2h}(w_3-w_4)^{-2h}F(x) \ ,$$

where $x = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$, and h is the conformal dimension of $\phi_{2,1}$.

The level-2 null-vector condition translates into the following partial differential equations:

$$\Bigg(\frac{3}{2(2h+1)}\partial_{w_i}^2 - \sum_{j\neq i}\Big(\frac{\partial_{w_j}}{(w_i-w_j)} + \frac{h}{(w_i-w_j)^2}\Big)\Bigg)G(\{w_i\}) = 0 \ .$$

a. Show, by considering the special values $w_1 = 0, w_2 = x, w_3 = \infty, w_4 = 1$, that $F(x)$ satisfies

$$\Bigg(x(1-x)\partial_x^2 + \Big(\frac{2(1-4h)}{3} + \frac{4(h-1)}{3}x\Big)\partial_x - \frac{2h(2h+1)}{3}\frac{x}{1-x}\Bigg)F(x) = 0$$

b. Show that the differential equation obtained in a. can be brought into the form of the hypergeometric differential equation, by defining a function $H(x) = f(x)F(x)$, such that $H(x)$ satisfies

$$\Bigg(x(1-x)\partial_x^2 + \Big(\frac{2(1-4h)}{3} + \frac{4(4h-1)}{3}x\Big)\partial_x + \frac{4h(1-4h)}{3}\Bigg)H(x) = 0 \qquad (1)$$

2. Show that in the case of the Ising model, two independent solutions of the differential equation (1) (with $h = \frac{1}{16}$) take the form

$$H^{\pm}(x) = \sqrt{\frac{1\pm\sqrt{1-x}}{2}} \ ,$$

by making a suitable (trigonometric) transformation.

Thus, the (chiral) four-point σ-correlator of the Ising model takes the form

$$\langle\sigma(w_1)\sigma(w_2)\sigma(w_3)\sigma(w_4)\rangle^{\pm} = (w_1-w_2)^{-1/8}(w_3-w_4)^{-1/8}(1-x)^{-1/8}\sqrt{\frac{1\pm\sqrt{1-x}}{2}} \ .$$

3. Majorana fermions with periodic and anti-periodic boundary conditions.

The mode expansion for free (Majorana) fermions reads $\psi(z) = \sum_n \psi_n z^{-n-1/2}$, or $\psi_n = \oint \frac{dz}{2\pi i} z^{n-1/2}\psi(z)$.

a. Show that the modes obey $\{\psi_n, \psi_m\} = \delta_{n+m,0}$.

We will now consider periodic and anti-periodic boundary conditions for the fermion $\psi(z)$ when z is moved around the origin: $\psi(e^{2\pi i}z) = \pm\psi(z)$. The modes n are half integer $n \in \mathbb{Z} + \frac{1}{2}$ in the periodic (P) case, and integer $n \in \mathbb{Z}$ in the anti-periodic (A) case.

b. Use the explicit mode expansions to show that

$$\langle\psi(z)\psi(w)\rangle_{\mathrm{P}} = \frac{1}{z-w}$$
$$\langle\psi(z)\psi(w)\rangle_{\mathrm{A}} = \frac{1}{2}\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w}$$

It is given that

$$\langle\sigma(w_1)\psi(z_1)\psi(z_2)\sigma(w_2)\rangle = \frac{1}{2}(w_1-w_2)^{-1/8}\frac{\left(\frac{(z_1-w_1)(z_2-w_2)}{(z_1-w_2)(z_2-w_1)}\right)^{1/2} + \left(\frac{(z_1-w_2)(z_2-w_1)}{(z_1-w_1)(z_2-w_2)}\right)^{1/2}}{(z_1-z_2)}$$

c. Consider $\langle\sigma(\infty)|\psi(z_1)\psi(z_2)|\sigma(0)\rangle$, where $\langle\sigma(\infty)| = \lim_{w\to\infty}\langle 0|\sigma(w)w^{2h}$, and argue that σ 'changes the boundary conditions on ψ'.