Exercises CFT-course fall 2023, set 7.

1. Consequences of null-vectors in the minimal models.

a. Show that a necessary condition for the (chiral) correlator of primaries

$$\langle \phi_{2,1}(z_1)\phi_{r,s}(z_2)\phi(z_3)\rangle$$

to be non-zero is that the scaling dimension h of $\phi(z_3)$ is equal to $h = h_{r\pm1,s}$, by making use of the level 2 null vector condition. Recall:

$$h_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} ,$$

where p and p' are relative prime and $0 < r < p'$ and $0 < s < p$.

Note that this condition $h = h_{r\pm1,s}$ is not sufficient in general, as the three point function might still vanish. The necessary and sufficient condition is expressed in terms of the 'fusion rules', which for $\phi_{r,1}$ read

$$\phi_{r,1} \times \phi_{r',s} = \sum_{\substack{k=|r-r'|+1 \\ k+r+r'=1 \text{ mod } 2}}^{k_{\max}} \phi_{k,s}$$

where $k_{\max} = \min(r + r' - 1, 2p' - r - r' - 1)$. These fusion rules express which three point functions are non-zero.

b. Ignoring the truncation in the upper limit $k_{\max}$, interpret the fusion rules for $\phi_{r,1}$ in terms of tensor products of $SU(2)$.

2. Non-unitarity of Virasoro representations with $0 < c < 1$.

The vanishing curves of the 'Kac determinant' are (for instance) given by

$$h_{r,s}(c) = \frac{1-c}{96}\left[\left((r+s) \pm (r-s)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] .$$

Given that the Kac determinant only has positive eigenvalues when $c > 1$ and $h > 0$, argue that unitary representations are excluded in the region $0 < c < 1$ and $h > 0$, except for possibly those points on the vanishing curves (hint: expand around $c = 1$).

3. The hypergeometric differential equation reads

$$\big(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\big) f(z) = 0$$

a. Find a solution around $z = 0$ by substituting $f(z) = \sum_{n\geq0} f_n z^n$, and express the result in terms of $(a)_n$, $(b)_n$ and $(c)_n$, where $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$, i.e. $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

Answer: $f(z) = F(a,b,c;z) = \sum_{n\geq0} \frac{(a)_n(b)_n}{(c)_n n!} z^n$. Note: the other solution around $z = 0$ reads $z^{1-c}F(a-c+1, b-c+1, 2-c; z)$.

b. When are polynomial solutions of the hypergeometric differential equation possible?