Solutions of nuclear physics tutorial 2

October 3, 2018

1 Shell model

These are cases where A is odd so you just need to look for the unpaired nucleus and read its l and J numbers. Parity is $\pi = (-1)^l$ and J corresponds to the nuclear spin. Indeed after the pairing hypothesis for ground-state nuclei, pairs of neutrons and pairs of protons in a given sub-shell always couple to give a combined angular momentum of zero, even when the sub-shell is not filled. Answers are collected in figure 1.

2 Deuteron : the simplest bound state

a) In the centre of mass, a state of energy E is described by a wave function ψ solution of the Schrödinger equation which is for this potential :

$$-\frac{\hbar^2}{2\mu}\,\psi''(r) - V_0\,\psi(r) = E\,\psi(r) \quad \text{for } r \in [0, R] \tag{1}$$

$$-\frac{\hbar^2}{2\mu}\,\psi''(r) = E\,\psi(r) \quad \text{for } r \in]R, +\infty[\tag{2}$$

where $\mu \triangleq \frac{m_p m_n}{m_p + m_n}$ is the reduced mass of the system. The above equations are more conveniently written as follows :

$$\psi''(r) + \frac{2\mu(V_0 + E)}{\hbar^2}\,\psi(r) = 0 \quad \text{for } r \in [0, R] \tag{3}$$

$$\psi''(r) + \frac{2\mu E}{\hbar^2}\,\psi(r) = 0 \quad \text{for } r \in]R, +\infty[\tag{4}$$

b) A bound state of a particle particle in a potential V is a state of energy E such that $E < \lim\limits_{x \to -\infty} V(x)$ AND $E < \lim\limits_{x \to +\infty} V(x)$. It means that the particle is trapped in the potential and cannot escape.
A scattering state of a particle particle in a potential V is a state of energy E such that $E > \lim\limits_{x \to -\infty} V(x)$ OR $E > \lim\limits_{x \to +\infty} V(x)$. It means that the particle can go away.

c) Coming back to our problem where the variable is constrained in the range $[0, +\infty[$, it is clear that a bound state is characterized by $E < 0$ and of course one also imposes $|E| < V_0$ so that $V_0 + E > 0$. Then, solutions of (5) and (6) are :

$$\psi(r) = A\ \sin(K_1 r) + B\ \cos(K_1 r) \quad \text{for } r \in [0, R] \tag{5}$$

$$\psi(r) = C\ e^{K_2 r} + D\ e^{-K_2 r} \quad \text{for } r \in]R, +\infty[\tag{6}$$

where $(A, B, C, D) \in \mathbb{R}^2$ will be determined soon and $K_1 \triangleq \sqrt{\frac{2\mu(V_0+E)}{\hbar^2}}$, $K_2 \triangleq \sqrt{-\frac{2\mu E}{\hbar^2}}$.

First of all, to prevent from divergence at infinity one has to impose $C = 0$. Then continuity of ψ and ψ' at $r = R$ yields :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7}$$

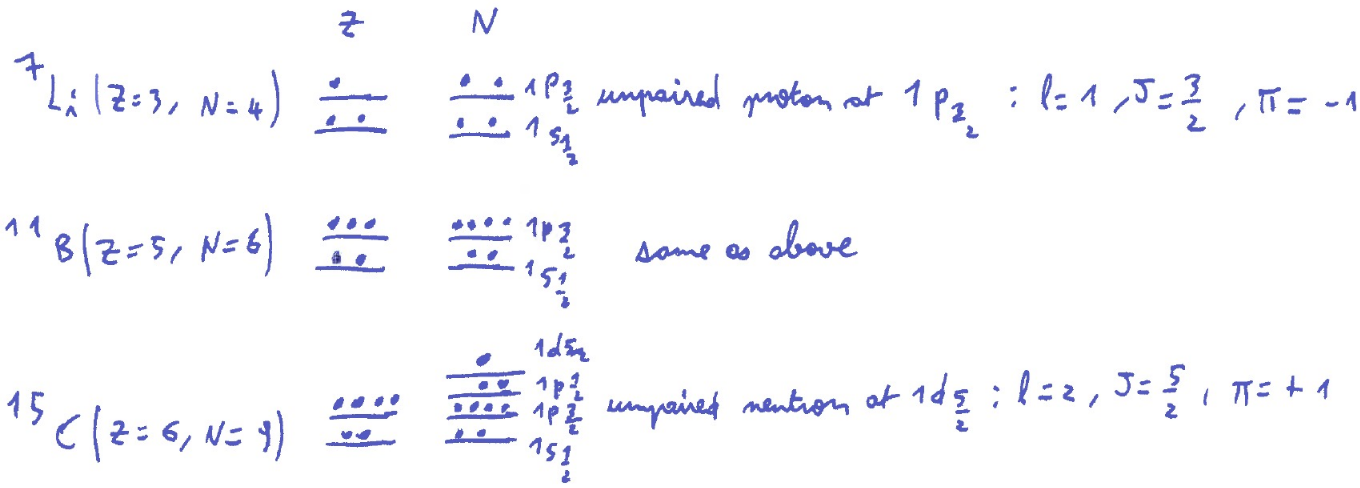

Figure 1: Shell model for ^7Li, ^{11}B and ^{15}C

$$\begin{cases} A\,\sin(K_1R) + B\,\cos(K_1R) = D\,e^{-K_2R} \\ A\,\cos(K_1R) - B\,\sin(K_1R) = -\frac{D}{K_2}\,e^{-K_2R} \end{cases} \Leftrightarrow \begin{pmatrix} \sin(K_1R) & \cos(K_1R) \\ \cos(K_1R) & -\sin(K_1R) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{K_2} \end{pmatrix} D\,e^{-K_2R}$$

The determinant of the system is $\Delta = -1$, then using Cramer's formulae one finds :

$$A = \frac{D\,e^{-K_2R}}{\Delta} \begin{vmatrix} 1 & \cos(K_1R) \\ -\frac{1}{K_2} & -\sin(K_1R) \end{vmatrix} = D\,e^{-K_2R}\left(\sin(K_1R) - \frac{\cos(K_1R)}{K_2}\right) \tag{8}$$

$$B = \frac{D\,e^{-K_2R}}{\Delta} \begin{vmatrix} \sin(K_1R) & 1 \\ \cos(K_1R) & -\frac{1}{K_2} \end{vmatrix} = D\,e^{-K_2R}\left(\cos(K_1R) + \frac{\sin(K_1R)}{K_2}\right) \tag{9}$$

We still need to determine D. Since $|\psi|^2$ is interpretating as a probability density, it has to be normalized i.e. $\int_0^{+\infty} |\psi(r)|^2 dr = 1$. For the sake of simplicity, let us write $A \equiv A_1D$ and $B \equiv B_1D$ with :

$$A_1 = e^{-K_2R}\left(\sin(K_1R) - \frac{\cos(K_1R)}{K_2}\right) \tag{10}$$

$$B_1 = e^{-K_2R}\left(\cos(K_1R) + \frac{\sin(K_1R)}{K_2}\right) \tag{11}$$

Then,

$$(A_1D)^2 \int_0^R \sin^2(K_1r)\,dr + (B_1D)^2 \int_R^{+\infty} e^{-2K_2r}\,dr = 1 \tag{12}$$

$$\Leftrightarrow \quad A_1^2 \int_0^R \frac{1-\cos(2K_1r)}{2}\,dr + B_1^2\left[-\frac{1}{2K_2}\,e^{-2K_2r}\right]_R^{+\infty} = \frac{1}{D^2} \tag{13}$$

$$\Leftrightarrow \quad A_1^2\left(\frac{R}{2} - \frac{\sin(2K_2R)}{4K_1}\right) + \frac{B_1^2}{2K_2}\,e^{-2K_2R} = \frac{1}{D^2} \tag{14}$$

which leads to the amazing expression :

$$D = \frac{e^{K_2R}}{\sqrt{\left(\sin(K_1R) - \frac{\cos(K_1R)}{K_2}\right)^2\left(\frac{R}{2} - \frac{\sin(2K_2R)}{4K_1}\right)^2 + \left(\cos(K_1R) + \frac{\sin(K_1R)}{K_2}\right)^2 e^{-2K_2R}}} \tag{15}$$

If you find a nicer way to write it, please email it to anthony.physth@gmail.com

Bonus : The given potential is actually the spherical square well potential, meaning that one should work with the Schrödinger equation in spherical coordinates. First step is to separate variables. Since V is only a function of r, equations for polar angle θ and azimuthal angle φ leads to the spherical harmonics $Y_m^l(\theta,\varphi)$. Solutions of the radial equation for $r < R$ are the spherical Bessel functions $j_l(K_1r)$ while one gets Hankel functions $I_{l+\frac{1}{2}}(K_2r)$ and $K_{l+\frac{1}{2}}(K_2r)$.

3 Fermi model

a) In the Fermi model, we assume that nucleons do not interact like particles in a gas and are trapped in a potential. Neutrons and protons are considered separately viz. like two different gas and it is clear in this question that you have to assume that they both have the same potential. Nucleons are fermions so two of them cannot occupy the same energy state (except if they have different spins) after Pauli's exclusion principle. Then they fill energy states from the the bottom of the potential to a top level which is by definition the Fermi level and has the Fermi energy E_F measured from a zero which does not necessarily fit with the bottom. One defines from it the Fermi momentum p_F which is such that $E_F = \frac{p_F^2}{2m}$ where m is the mass of one nucleon. In this question, no distinction is made between protons and neutrons ; anyway, they have almost the same mass $938\ MeV$. The Fermi sphere is the sphere of radius p_F in the momentum space ; this surface is very important in hard condensed matter.
If the number of nucleons is sufficiently big, one can consider a continuum of energy states ; basically, quantization of energy is not perceptible for macroscopic systems like gas. The number of states with energy between ϵ and $\epsilon + d\epsilon$ is written as $\rho(\epsilon)d\epsilon$ where $\rho(\epsilon)$ is the density of states computed in appendix A of the book. Finally, using tools of statistical physics and several assumptions (thermodynamic limit, very low temperature) one can derive the formula given in the question :

$$\frac{A}{2} = \frac{2V}{h^3}\int d^3p \quad \text{where integration is performed over the Fermi sphere} \tag{16}$$

where the integral is nothing but the volume of the Fermi sphere i.e. $\frac{4}{3}\pi p_F^3$ and V is the volume of the nucleus which we supppose spherical with radius $R = R_0\ A^{\frac{1}{3}}$, $R_0 = 1.2\ fm$. Then one gets :

$$\frac{A}{2} = \frac{2}{(2\pi\hbar)^3}\,\frac{4}{3}\pi R_0^3\,A\,\frac{4}{3}\pi p_F^3 \quad\Leftrightarrow\quad p_F = 3^{\frac{2}{3}}\pi^{\frac{1}{3}}\,\frac{\hbar}{2R_0} \quad\Rightarrow\quad E_F = 3^{\frac{4}{3}}\pi^{\frac{2}{3}}\,\frac{(\hbar c)^2}{8R_0^2\,mc^2} = 3^{\frac{4}{3}}\pi^{\frac{2}{3}}\,\frac{197^2}{8\times 1.2^2\times 938} \simeq 33\,MeV$$

where we use $\hbar c = 197\ MeV.fm$.

b) First case : protons and neutrons have the same potential but $Z \neq N$. Due the rules of filling energy states, it is clear that protons and neutrons will have different Fermi levels. Expressed in a childy way, you stack balls in two identical boxes then if you don't put the name number of balls in them, the heights of balls are different.

Second case : $Z \neq N$ and protons and neutrons do not feel the same potential. Neutrons are trapped in a square well potential. For protons, this square well potential is deformed by Coulomb interaction and possibly do not have the same minimum (see figure 7.2 of the book, also reproduced in the slides of the lecture). The difference of Fermi levels is even more obvious. Extending the analogy of balls, in this case the two boxes have different shapes and do not stand at the same altitude.

c) We first need to compute the Fermi momentum of protons and neutrons which are now different. Calculations are almost identical to the one done in question a) and yield :

$$P_F(protons) = \left(\frac{\pi Z}{A}\right)^{\frac{1}{3}}\left(\frac{3}{2}\right)^{\frac{2}{3}}\frac{\hbar}{R_0} \tag{17}$$

$$P_F(neutrons) = \left(\frac{\pi N}{A}\right)^{\frac{1}{3}}\left(\frac{3}{2}\right)^{\frac{2}{3}}\frac{\hbar}{R_0} \tag{18}$$

We now compute the average kinetic energy with the given formula (coming from statistical physics) :

$$E = \frac{2}{(2\pi\hbar)^3}\,\frac{4}{3}\pi R_0^3\,A\,\frac{4\pi}{2m_p}\int_0^{P_F(protons)} p^4dp + \frac{2}{(2\pi\hbar)^3}\,\frac{4}{3}\pi R_0^3\,A\,\frac{4\pi}{2m_n}\int_0^{P_F(neutrons)} p^4dp \tag{19}$$

$$= \frac{2A}{3\pi}\left(\frac{R_0}{\hbar}\right)^3\left\{\frac{P_F^5(protons)}{5m_p} + \frac{P_F^5(neutrons)}{5m_n}\right\} = \frac{1}{5}\left(\frac{3}{2}\right)^{\frac{7}{3}}\left(\frac{\hbar}{R_0}\right)^2\left(\frac{\pi}{A}\right)^{\frac{2}{3}}\left\{\frac{Z^{\frac{5}{3}}}{m_p} + \frac{N^{\frac{5}{3}}}{m_n}\right\} \tag{20}$$

This is nothing but formula (7.10) of the book written using prime number decomposition of the numerical prefactor.