Solutions of nuclear physics tutorial 3

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1 Activity of a physics student

a) We know that the molar mass $M = 12\ g/mol$ is the mass of $N_A = 6.02 \times 10^{23}$ atoms ^{12}C so the mass in g of one atom ^{12}C is $\frac{M}{N_A}$. Subsequently :

$$1\ u = \frac{1}{12}\frac{M}{N_A}\,10^{-3}Kg = \frac{10^{-3}}{6.02 \times 10^{23}}\,\frac{(3 \times 10^8)^2}{1.6 \times 10^{-13}}\,MeV/c^2 = 934.3\ MeV/c^2 \tag{1}$$

One should take values of c, N_A and e with more digits to get the official $931.5\ MeV/c^2$.

b) The number N_C of atoms ^{14}C is equal to the ratio of the mass of carbon 14 in the body by the mass of one atom ^{14}C. The latter is approximately $14\ u$ while the mass of carbon 14 in the body is equal to the mass of the body (70 Kg) times the fraction of ^{14}C. This fraction is nothing but the percentage of carbon in human body times the natural abundance of ^{14}C. In the end,

$$N_C = \frac{0.18 \times 10^{-12} \times 70 \times 6.02 \times 10^{23}}{14 \times 10^{-3}} = 5.4 \times 10^{14} \tag{2}$$

The number N_K of atoms ^{40}K is obtained similarly :

$$N_K = \frac{0.002 \times 1.2 \times 10^{-4} \times 70 \times 6.02 \times 10^{23}}{40 \times 10^{-3}} = 2.5 \times 10^{20} \tag{3}$$

c) We recall that the activity of a radioactive source is $A(t) = \left|\frac{dN}{dt}\right| \equiv \lambda\ N(t)$ since the number of radioactive atoms decreases according the exponential law of parameter λ (probability per unit time). Half-life time $\tau_{\frac{1}{2}}$ are given so one can infer this parameter : $\lambda = \frac{\ln(2)}{\tau_{\frac{1}{2}}}$.

$$A_C = \frac{5.4 \times 10^{14} \times \ln(2)}{5730 \times 365.25 \times 24 \times 3600} = 2070\ Bq \tag{4}$$

$$A_K = \frac{2.5 \times 10^{20} \times \ln(2)}{1.25 \times 10^9 \times 365.25 \times 24 \times 3600} = 4390\ Bq \tag{5}$$

2 Gamma ray flux

a) We remind that the activity is the number of decays per second and one can see that two photons are produced at each decay. We want the flux Φ at a distance $1\ m$ from the source. Assuming that the emission of photons is spherically symmetric, the flux is the ratio of the rate of photons produced by the surface of the unit sphere :

$$\Phi = \frac{2A}{4\pi} = 4.42 \times 10^7\ photons\ .\ s^{-1}.m^{-2} \tag{6}$$

b) The half-value layer is the thickness of the material at which the intensity of radiation entering it is reduced by one half. For gamma rays, it varies with the nature of the material from a few millimeters to a few centimeters.

3 Study of a chain reaction

a) Equations describing the chain reaction are :

$$\frac{dN_{Cs}}{dt} = -\lambda_{Cs}\, N_{Cs}(t) \quad (7)$$
$$\frac{dN_{Ba}}{dt} = \lambda_{Cs}\, N_{Cs}(t) - \lambda_{Ba}\, N_{Ba}(t) \quad (8)$$
$$\frac{dN_{La}}{dt} = \lambda_{Ba}\, N_{Ba}(t) \quad (9)$$

with initial conditions $N_{Cs}(t=0) = N_0$, $N_{Ba}(t=0) = 0$ and $N_{La}(t=0) = 0$. The value of N_0 is not given but one knows the initial activity $A_{Cs}(t=0) = \lambda_{Cs}\, N_0$. From the half-life time, we can infer :

$$N_0 = \frac{A_{Cs}(t=0)\, \tau_{\frac{1}{2}}}{\ln(2)} = \frac{37 \times 10^6 \times 9.5 \times 60}{\ln(2)} = 3.04 \times 10^{10} \quad (10)$$

Where we used $1\ Cu = 37\ GBq$. One also finds $\lambda_{Cs} = 7.3 \times 10^{-2}\ min^{-1}$ and $\lambda_{Ba} = 8.3 \times 10^{-3}\ min^{-1}$.

b) The activity of ^{139}Ba is $A_{Ba} = \lambda_{Ba}\, N_{Ba}$ so we have to solve (8).

Solution of (7) is $N_{Cs}(t) = N_0\, e^{-\lambda_{Cs} t}$, we plug it in (8) :

$$\frac{dN_{Ba}}{dt} + \lambda_{Ba}\, N_{Ba}(t) = \lambda_{Cs}\, N_0\, e^{-\lambda_{Cs} t} \quad (11)$$

Integrating factor is $\alpha(t) = e^{\lambda_{Ba} t}$ so that :

$$\frac{d\big(N_{Ba}(t)\alpha(t)\big)}{dt} = \alpha(t)\lambda_{Cs}\, N_0\, e^{-\lambda_{Cs} t} \quad \Leftrightarrow \quad N_{Ba}(t)\alpha(t) = \lambda_{Cs}\, N_0 \int^t \alpha(t')\, e^{-\lambda_{Cs} t'} dt' + C \quad (12)$$

where $C \in \mathbb{R}$ will be determined by initial conditions.

$$N_{Ba}(t) = \lambda_{Cs}\, N_0\, e^{-\lambda_{Ba} t} \int^t e^{(\lambda_{Ba} - \lambda_{Cs})t'} dt' + C\, e^{-\lambda_{Ba} t} = \frac{\lambda_{Cs}\, N_0}{\lambda_{Ba} - \lambda_{Cs}}\, e^{-\lambda_{Ba} t}\, e^{(\lambda_{Ba} - \lambda_{Cs})t} + C\, e^{-\lambda_{Ba} t} \quad (13)$$

From $N_{Ba}(t=0) = 0$, one gets $C = -\frac{\lambda_{Cs}\, N_0}{\lambda_{Ba} - \lambda_{Cs}}$ and finally :

$$N_{Ba}(t) = \frac{\lambda_{Cs}\, N_0}{\lambda_{Ba} - \lambda_{Cs}} \Big(e^{-\lambda_{Cs} t} - e^{-\lambda_{Ba} t}\Big) = \frac{A_{Cs}(t=0)}{\lambda_{Ba} - \lambda_{Cs}} \Big(e^{-\lambda_{Cs} t} - e^{-\lambda_{Ba} t}\Big) \quad (14)$$

Then,

$$A_{Ba}(t) = \frac{\lambda_{Ba}\, A_{Cs}(t=0)}{\lambda_{Ba} - \lambda_{Cs}} \Big(e^{-\lambda_{Cs} t} - e^{-\lambda_{Ba} t}\Big) \quad (15)$$

Note we were not even obliged to calculate N_0 since this expression directly depends on the initial activity.
To find the maximum of A_{Ba}, we need to look for zeros of its derivative.

$$\frac{dA_{Ba}}{dt} \propto -\lambda_{Cs}\, e^{-\lambda_{Cs} t} + \lambda_{Ba}\, e^{-\lambda_{Ba} t} \quad (16)$$
$$-\lambda_{Cs}\, e^{-\lambda_{Cs} t} + \lambda_{Ba}\, e^{-\lambda_{Ba} t} = 0 \quad \Leftrightarrow \quad e^{(\lambda_{Cs} - \lambda_{Ba})t} = \frac{\lambda_{Cs}}{\lambda_{Ba}} \quad (17)$$

A_{Ba} has a maximum at :

$$t_m = \frac{\ln\left(\frac{\lambda_{Cs}}{\lambda_{Ba}}\right)}{\lambda_{Cs} - \lambda_{Ba}} = 33.6\ min \quad (18)$$

Using (), one eventually obtains $A_{max} = 8.6 \times 10^{-5}\ Cu$.

c) One needs to solve (9) :

$$\frac{dN_{La}}{dt} = \frac{\lambda_{Ba}\, A_{Cs}(t=0)}{\lambda_{Ba} - \lambda_{Cs}} \Big(e^{-\lambda_{Ba} t} - e^{-\lambda_{Cs} t}\Big) \quad (19)$$

$$\Leftrightarrow \quad N_{La}(t) = \frac{\lambda_{Ba}\, A_{Cs}(t=0)}{\lambda_{Ba} - \lambda_{Cs}} \int_0^t \Big(e^{-\lambda_{Ba} t'} - e^{-\lambda_{Cs} t'}\Big)\, dt' = \frac{\lambda_{Ba}\, A_{Cs}(t=0)}{\lambda_{Ba} - \lambda_{Cs}} \left(\frac{e^{-\lambda_{Cs} t} - 1}{\lambda_{Cs}} - \frac{e^{-\lambda_{Ba} t} - 1}{\lambda_{Ba}}\right)$$

After one hour, there are $N_{La}(t = 60\ min) = 9.6 \times 10^9$ atoms ^{139}La. Notice that one has to be careful with units in this numerical application. One way to prevent mistakes is to write $A_{Cs}(t = 0) = \lambda_{Cs}\ N_0$, use the value (10) of N_0 and express everywhere the constants λ_{Cs} and λ_{Ba} in min^{-1}.