Solutions of nuclear physics tutorial 1

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1 Form factor

a) Spherical symmetry means $\rho(\boldsymbol{r}) = \rho(r)$. We have taken $\hbar = 1$ in the definition of the form factor.

$$F(\boldsymbol{q}) = \frac{1}{Ze}\int_{\mathbb{R}^3}\rho(r)\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}d^3r = \frac{1}{Ze}\int_{\varphi=0}^{2\pi}d\varphi\int_{r=0}^{+\infty}\int_{\theta=0}^{\pi}\rho(\boldsymbol{r})\, e^{iqr\cos(\theta)}\sin(\theta)\, r^2\, d\theta\, dr \tag{1}$$

$$= \frac{2\pi}{Ze}\int_{r=0}^{+\infty}\rho(r)\Big[-\frac{1}{iq}\, e^{iqr\cos(\theta)}\Big]_{\theta=0}^{\pi} rdr = \frac{4\pi}{Ze}\int_0^{+\infty}\frac{\rho(r)}{q}\,\frac{e^{iqr}-e^{-iqr}}{2i}\, rdr \tag{2}$$

$$\equiv \frac{4\pi}{Ze}\int_0^{+\infty}\rho(r)\,\frac{\sin(qr)}{q}\, rdr \tag{3}$$

b) The Heaviside step function is such that $\theta(x) = 1$ for $x \geqslant 1$ and $\theta(x) = 0$ for $x < 1$.

i) $\rho(\boldsymbol{r}) = \rho_0\,\theta(R-r)$ only depends on r so we can use equation (3). Moreover, $\rho(r) = \rho_0$ for $r \in [0, R]$ and $\rho(r) = 0$ for $r > R$.

$$F(\boldsymbol{q}) = \frac{4\pi}{Ze}\int_0^R\rho_0\,\frac{\sin(qr)}{q}\, rdr = \frac{4\pi}{Ze}\rho_0\left(\Big[-\frac{r}{q}\,\cos(qr)\Big]_0^R + \int_0^R\frac{\cos(qr)}{q}\, dr\right) \tag{4}$$

$$= \frac{4\pi}{Ze}\rho_0\left(-\frac{R}{q}\,\cos(qR) + \Big[\frac{\sin(qr)}{q^2}\Big]_0^R\right) = \frac{4\pi}{Ze}\,\frac{\rho_0}{q^3}\Big(\sin(qR) - qR\cos(qR)\Big) \tag{5}$$

ii) $\rho(\boldsymbol{r}) = \rho_0\, e^{-\ln(2)\frac{r^2}{R^2}}$ only depends on r so we COULD use equation (3) but since it is a Gaussian, it is better to apply the definition with a trick. We work in Cartesian coordinates $\boldsymbol{r} = (x_1, x_2, x_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$.

$$F(\boldsymbol{q}) = \frac{1}{Ze}\int_{\mathbb{R}^3}\rho_0\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}-\ln(2)\frac{r^2}{R^2}}\, d^3r = \frac{\rho_0}{Ze}\int_{\mathbb{R}^3}e^{i(q_1x_1+q_2x_2+q_3x_3)-\frac{\ln(2)}{R^2}\left(x_1^2+x_2^2+x_3^2\right)}\, dx_1dx_2dx_3 \tag{6}$$

$$= \frac{\rho_0}{Ze}\prod_{j=1}^{3}\int_{-\infty}^{+\infty}e^{iq_jx_j-\frac{\ln(2)}{R^2}x_j^2}\, dx_j \tag{7}$$

A second order polynomial is exponentiated. We put it in its canonical form :

$$i\, q_jx_j - \frac{\ln(2)}{R^2}x_j^2 = -\frac{\ln(2)}{R^2}\left(x_j^2 - i\,\frac{qR^2x_j}{\ln(2)}\right) \equiv -\frac{\ln(2)}{R^2}\left[\left(x_j - i\frac{qR^2}{2\ln(2)}\right)^2 - \left(i\frac{qR^2}{2\ln(2)}\right)^2\right] \tag{8}$$

We plug this expression in the integral of equation (7) and call it I_j :

$$I_j = \int_{-\infty}^{+\infty}e^{iq_jx_j-\frac{\ln(2)}{R^2}x_j^2}\, dx_j = e^{-\frac{q_j^2R^2}{4\ln(2}}\int_{-\infty}^{+\infty}e^{-\frac{\ln(2)}{R^2}\left(x_j-i\frac{qR^2}{2\ln(2)}\right)^2}dx_j \equiv e^{-\frac{q_j^2R^2}{4\ln(2)}}\int_{-\infty}^{+\infty}e^{-\frac{\ln(2)}{R^2}x_j^2}dx_j \tag{9}$$

The last step is rigorously justified by writing the integral in the complex plane and applying Cauchy theorem to an appropriate contour. A simpler argument is the change of variable $x'_j = x_j - i\frac{qR^2}{2\ln(2)}$.

$$I_j = e^{-\frac{q_j^2 R^2}{4\ln(2)}} \int_{-\infty}^{+\infty} e^{-u^2} \frac{R}{} \, du \equiv R\sqrt{\frac{\pi}{\ln(2)}}\, e^{-\frac{q_j^2 R^2}{4\ln(2)}} \tag{10}$$

We have performed the change of variables $u = \sqrt{\ln(2)}\frac{x_j}{R}$. We eventually infer :

$$F(\boldsymbol{q}) = \frac{\rho_0}{Ze} R^3 \left(\frac{\pi}{\ln(2)}\right)^{\frac{3}{2}} e^{-\frac{q^2 R^2}{4\ln(2)}} \tag{11}$$

where we made use once more of $q^2 = q_1^2 + q_2^2 + q_3^2$.

2 Fermi distribution

a) The shape of the curve strongly depends on the ratio R/a. See figure 1.

$$\rho(\boldsymbol{r}) = \frac{\rho_0}{1 + e^{\frac{r-R}{a}}} \tag{12}$$

For $R \gg a$, one has $\rho(r=0) \simeq \rho_0$. Furthermore, $\lim_{a\to 0} \frac{\rho_0}{1+e^{\frac{r-R}{a}}} = \rho_0\, \theta(R-r)$ as you can see on the blue curve where for wich $a = 10^{-2}$.

b) Let us show that this distribution is symmetric with respect to the point $(R, \frac{\rho_0}{2})$ which means :

$$\forall r \in \mathbb{R}^+, \quad \frac{\rho(R+r) + \rho(R-r)}{2} = \frac{\rho_0}{2} \tag{13}$$

Derivation goes like this :

$$\forall r \in \mathbb{R}^+, \quad \rho(R+r) + \rho(R-r) = \frac{\rho_0}{1+e^{\frac{r}{a}}} + \frac{\rho_0}{1+e^{\frac{-r}{a}}} = \frac{\rho_0}{1+e^{\frac{r}{a}}} + \frac{\rho_0\, e^{\frac{r}{a}}}{e^{\frac{r}{a}}+1} = \rho_0\, \frac{1+e^{\frac{r}{a}}}{e^{\frac{r}{a}}+1} \equiv \rho_0 \tag{14}$$

Let us show that this distribution has an inflection point i.e. a point of abscissa r_0 such that $\rho''(r_0) = 0$ where prime means derivative.

$$\rho'(r) = -\frac{\frac{\rho_0}{a}\, e^{\frac{r-R}{a}}}{\left(1+e^{\frac{r-R}{a}}\right)^2} \tag{15}$$

$$\rho''(r) = \frac{-\frac{\rho_0}{a^2}\, e^{\frac{r-R}{a}} \left(1+e^{\frac{r-R}{a}}\right)^2 + \frac{2\rho_0}{a^2}\, e^{\frac{r-R}{a}} \left(1+e^{\frac{r-R}{a}}\right)}{\left(1+e^{\frac{r-R}{a}}\right)^4} = \frac{\frac{\rho_0}{a^2}\, e^{\frac{r-R}{a}} \left(1-e^{\frac{r-R}{a}}\right)}{\left(1+e^{\frac{r-R}{a}}\right)^3} \tag{16}$$

Clearly the only solution r_0 of the equation $\rho''(r) = 0$ is $r_0 = R$. Hence, the point $(R, \frac{\rho_0}{2})$ is an inflection point.

Note : Not all symmetric points are inflection point and not all inflection points are symmetric points of the whole curve (they can be only locally symmetric).

c) The skin thickness is the distance $t = |r_1 - r_2|$ such that $\rho(r_1) = 0.9\,\rho_0$ and $\rho(r_2) = 0.1\,\rho_0$. Then,

$$\begin{cases} \frac{1}{1+e^{\frac{r_1-R}{a}}} = \frac{9}{10} \\ \frac{1}{1+e^{\frac{r_2-R}{a}}} = \frac{1}{10} \end{cases} \Leftrightarrow \begin{cases} e^{\frac{r_1-R}{a}} = \frac{1}{9} \\ e^{\frac{r_2-R}{a}} = 10 \end{cases} \Leftrightarrow \begin{cases} r_1 = -a\ \ln(9) + R \\ r_2 = a\ \ln(10) + R \end{cases} \tag{17}$$

So $t = r_2 - r_1 = a\left(\ln(10) + \ln(9)\right) = a\ \ln(90) = 2.9\, fm$. We infer $a = 0.64\, fm$.

3 Semi-empirical formulae

a) We look for the potential energy of a sphere of radius R with uniform charge distribution. First of all, we remind that the potential energy of two pointwise charges q_1 and q_2 separated by a distance r is :

$$E_{pot}(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \tag{18}$$

We now see the sphere as a superposition of infinitesemal shells. The shell of radius r and thickness dr has a charge $dq(r)$; its interior is the sphere of radius r and contains the charge $q(r)$. We stress on the fact that equation (18) gives the energy of interaction between charges q_1 and q_2. In the case of the sphere, the total potential energy is the sum of the energies of interaction between each shell with its interior so that it can be written as follows :

$$E(R) = \int_0^R \frac{q(r)dq(r)}{4\pi\epsilon_0 r} \tag{19}$$

Then we need to compute the charge of a shell of radius r and thickness dr, and also the charge $q(r)$ of its interior. However, the sphere has a uniform distribution of charge ρ which is nothing but the ratio of its total charge Q and its volume :

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \tag{20}$$

The charge of the shell of radius r and thickness dr is given by the product of this density by its volume :

$$dq(r) = \frac{Q}{\frac{4}{3}\pi R^3}\, 4\pi\, r^2 dr = \frac{3Q}{R^3}\, r^2 dr \tag{21}$$

The charge of the interior of this shell, which is the sphere of radius r, is simply the product of the density by its volume :

$$q(r) = \frac{Q}{\frac{4}{3}\pi R^3}\, \frac{4}{3}\pi r^3 = Q\left(\frac{r}{R}\right)^3 \tag{22}$$

It just remains to plug expressions (21) and (22) into equation (19) :

$$E(R) = \frac{3Q^2}{4\pi\epsilon_0 R^6} \int_0^R r^4 dr = \frac{3Q^2}{4\pi\epsilon_0 R^6} \left[r^5 5 \right]_0^R = \frac{3}{5}\, \frac{Q^2}{4\pi\epsilon_0 R} \tag{23}$$

b) The nuclear radius of a nucleus ${}^A_Z X$ is given by the semi-empirical formula $R = r_0\, A^{\frac{1}{3}}$ with $r_0 = 1.2\ fm$. Subsequently, the Coulomb energy of a nucleus ${}^A_Z X$ is :

$$E(A,Z) = \frac{3}{5}\, \frac{Z^2 e^2}{4\pi\epsilon_0 r_0\, A^{\frac{1}{3}}} \tag{24}$$

For numerical applications, it is interesting to make appear the fine structure constant $\alpha \triangleq \frac{e^2}{4\pi\epsilon_0 \hbar c} \simeq \frac{1}{137}$:

$$E(A,Z) = \frac{3}{5}\, \frac{Z^2 \alpha \hbar c}{r_0\, A^{\frac{1}{3}}} \tag{25}$$

Use $\hbar c = 197\ Mev.fm$ for numerical applications and see table 1.

$$E(A,Z) = \frac{3 \times 197}{5 \times 137 \times 1.2}\, \frac{Z^2}{A^{\frac{1}{3}}} = 0.72\, \frac{Z^2}{A^{\frac{1}{3}}}\ MeV \tag{26}$$

Note : we have found back the third term of the semi-empirical mass formula with an error 0.02.

c) The binding energy is given by Bethe Weizsäcker formula from the liquid droplet model :

$$B(A,Z) = 15.6\ A - 17.2\ A^{\frac{2}{3}} - 0.70\, \frac{Z^2}{A^{\frac{1}{3}}} - 23.3\, \frac{(A-2Z)^2}{A} + \delta(A,Z)\ MeV \tag{27}$$

where the pairing term is :

$$\delta(A,Z) = \begin{cases} -12\ A^{-\frac{1}{2}} & \text{if } Z \text{ and } N \text{ are both even} \\ 0 & \text{if } A \text{ is odd} \\ 12\ A^{-\frac{1}{2}} & \text{if } Z \text{ and } N \text{ are both odd} \end{cases} \tag{28}$$

See table 1 for numerical applications.

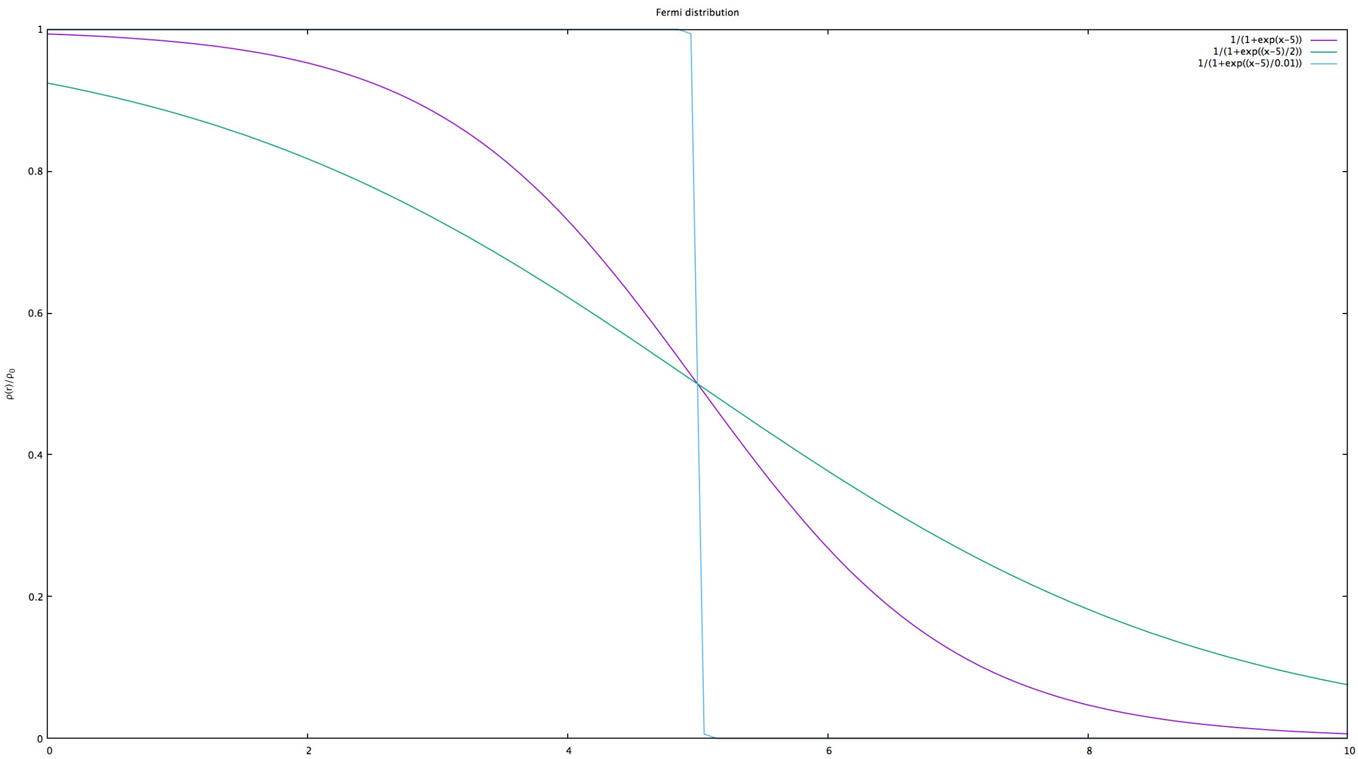

Figure 1: Plot of the Fermi distribution for $R = 5$ and $a \in \{1, 2, 0.01\}$

	${}^{21}_{10}Ne$	${}^{57}_{26}Fe$	${}^{209}_{83}Bi$
E(A,Z) in MeV	26.1	126.5	835.8
B(A,Z) in MeV	170.2	501.3	1635.9

Table 1: Coulomb and binding energies for ${}^{21}_{10}Ne$, ${}^{57}_{26}Fe$ and ${}^{209}_{83}Bi$