Solutions of the tutorial on Rutherford scattering

September 18, 2018

1 Relation between the impact parameter and the scattering angle

1) Let be $\boldsymbol{r}$ and $\boldsymbol{v}$ the position and velocity vectors of the alpha particle viz. $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$. We work in polar coordinates (r, ϕ) so that $\boldsymbol{r} = r\ \boldsymbol{e}_r$. Newton's second law for the alpha particle yields :

$$m\ \frac{d\boldsymbol{v}}{dt} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\ \boldsymbol{e}_r \tag{1}$$

There is a repulsion due to the electrostatic force.
We can assume that the nucleus is fixed since it is much heavier than the alpha particle. Otherwise, we should go into the centre of mass frame to study the relative motion of two objects with masses m_1 and m_2 moving simultaneously and introduce a fictive particle with the so-called reduced mass $\mu \triangleq \frac{m_1 m_2}{m_1+m_2}$. It is obvious that for $m_1 \gg m_2$ one has $\mu \simeq m_2$ like it is the case in the Rutherford scattering.

2) By definition, the orbital momentum of the alpha particle is $\boldsymbol{J} = m\ \boldsymbol{r} \times \boldsymbol{v}$. The electrostic force is a central force i.e. its norm only depends on the distance between the two interacting particle. You should know from your classical mechanics class that in such cases, the orbital momentum is conserved. A quick derivation goes as follows :

$$\frac{d\boldsymbol{J}}{dt} = m\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{v} + \boldsymbol{r} \times m\frac{d\boldsymbol{v}}{dt} \equiv m\ \boldsymbol{v} \times \boldsymbol{v} + r\ \boldsymbol{e}_r \times \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\ \boldsymbol{e}_r = \boldsymbol{0} \tag{2}$$

Since the orbital momentum is constant, we can identify it with the initial orbital momentum $\boldsymbol{J}_0$. In the Rutherford experiment, the alpha particle comes from afar (basically from a distance d sufficiently big so that there is no interaction) with impact parameter b and an initial velocity $\boldsymbol{v}_0 = v_0\ \boldsymbol{e}_x$. So $\boldsymbol{J}_0 = m\ \boldsymbol{OM}_0 \times \boldsymbol{v}_0$ where M_0 is a point with Cartesian coordinates $(-d, b)$. So $\boldsymbol{OM}_0 = b\ \boldsymbol{e}_y - d\ \boldsymbol{e}_x$ and subsequently :

$$\boldsymbol{J}_0 = m\ \big(b\ \boldsymbol{e}_y - d\ \boldsymbol{e}_x\big) \times v_0\ \boldsymbol{e}_x = mbv_0\ \boldsymbol{e}_y \times \boldsymbol{e}_x \equiv -mbv_0\ \boldsymbol{e}_z \tag{3}$$

where $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ is a direct orthogonal basis. Therefore $\boldsymbol{J} = J\ \boldsymbol{e}_z$ with $J = -mbv_0$.

3) We have just shown that the orbital momentum is constant and orthogonal to the (x, y)-plane. However, by definition of the cross product $\boldsymbol{J}$ is orthogonal to $\boldsymbol{r}$ and to $\boldsymbol{v}$. This means that both $\boldsymbol{r}$ and $\boldsymbol{v}$ are in the (x, y)-plane. Since the trajectory of the alpha particle is completely determined by $\boldsymbol{r}(t)$ and to $\boldsymbol{v}(t)$ for $t \geqslant t_0$, one concludes that the motion is constrained in the (x, y)-plane.

Let us project equation (1) on y-axis :

$$m\ \frac{dv_y}{dt} = \frac{1}{4\pi\epsilon_0}\ \frac{qQ}{r(t)^2}\ \boldsymbol{e}_r\big(\phi(t)\big) \cdot \boldsymbol{e}_y \quad \Leftrightarrow \quad dv_y = \frac{1}{4\pi\epsilon_0 m}\ \frac{qQ}{r(t)^2}\ \sin\big(\phi(t)\big)dt \tag{4}$$

We need a little trick to remove the trick the time dependence. Let us calculate the general expression of the orbital momentum in polar coordinates :

$$\boldsymbol{r}(t) = r(t)\ \boldsymbol{e}_r \tag{5}$$

$$\boldsymbol{v}(t) = \frac{dr}{dt}\ \boldsymbol{e}_r + r(t)\frac{d\phi}{dt}\ \boldsymbol{e}_\phi \tag{6}$$

$$\boldsymbol{J}(t) = m\ r(t)^2\ \frac{d\phi}{dt}\ \boldsymbol{e}_z \tag{7}$$

Using the result of the previous question, one makes the identification :

$$\forall t \geqslant t_0, \quad m\, r(t)^2\, \frac{d\phi}{dt} = -mbv_0 \quad \Leftrightarrow \quad d\phi = -\frac{bv_0}{r(t)^2}\, dt \tag{8}$$

We now plug this expression into equation (4) :

$$dv_y = -\frac{qQ}{4\pi\epsilon_0 mbv_0}\, \sin(\phi)d\phi \tag{9}$$

which is the required ODE with variables already separated.

4) We want to integrate this ODE along the trajectory viz. for $t \in [t_0, +\infty[$. To do this, we need to set the boundaries properly. It is very clear that the polar angle ϕ goes from π to θ. The velocity vector goes from $\boldsymbol{v}_0$, which has no y-component, to $\boldsymbol{v}_\infty$ which has to be determined. This is an elastic scattering meaning the energy and momentum are conserved.

In general, for elastic scattering of particles a and b conservation of energy and momentum is expressed as follows :

$$\begin{cases} \{m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b\}_{before} = \{m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b\}_{after} \\ \{\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2\}_{before} = \{\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2\}_{after} \end{cases} \tag{10}$$

Let us apply these conservation's laws to our situation : say a is the alpha particle and b the nucleus initially at rest, with the assumption $m_a \gg m_b$.

$$\begin{cases} \{m_a \boldsymbol{v}_a\}_{before} = \{m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b\}_{after} \\ \{\frac{1}{2} m_a v_a^2\}_{before} = \{\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2\}_{after} \end{cases} \quad \Leftrightarrow \quad \begin{cases} \{m_a \boldsymbol{v}_a\}_{before} = \{m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b\}_{after} \\ \{v_a^2\}_{before} = \{v_a^2 + \frac{m_b}{m_a} v_b^2\}_{after} \end{cases} \tag{11}$$

We have used the fact that masses of particles are invariant. Since $\frac{m_b}{m_a} \ll 1$, one gets at first order :

$$\begin{cases} \{m_a \boldsymbol{v}_a\}_{before} = \{m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b\}_{after} \\ \{v_a^2\}_{before} = \{v_a^2\}_{after} \end{cases} \tag{12}$$

NB : This approximation makes sense and it is possible to find $\{\boldsymbol{v}_a\}_{after}$ and $\{\boldsymbol{v}_b\}_{after}$ satisfying those relations.

Coming back to our original notations, the second line of (12) gives $||\boldsymbol{v}_\infty|| = ||\boldsymbol{v}_0||$. However, by definition of the scattering angle $\boldsymbol{v}_\infty$ has to be colinear to $\boldsymbol{e}_\theta$. We then infer :

$$\boldsymbol{v}_\infty = v_0\, \boldsymbol{e}_\theta \quad \Rightarrow \quad v_{y\infty} = v_0\, \boldsymbol{e}_\theta \cdot \boldsymbol{e}_y = v_0 \sin(\theta) \tag{13}$$

Integration of equation (9) :

$$\int_0^{v_0 \sin(\theta)} dv_y = -\frac{qQ}{4\pi\epsilon_0 mbv_0} \int_\pi^\theta \sin(\phi)d\phi \quad \Leftrightarrow \quad v_0 \sin(\theta) = -\frac{qQ}{4\pi\epsilon_0 mbv_0} \big[-\cos(\phi)\big]_\pi^\theta = \frac{qQ}{4\pi\epsilon_0 mbv_0} \big(\cos(\theta) + 1\big)$$

Use the well-known trigonometric relations

$$\begin{cases} \sin(\theta) = \frac{2\tan(\frac{\theta}{2})}{1+\tan^2(\frac{\theta}{2})} \\ \cos(\theta) = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2(\frac{\theta}{2})} \end{cases} \tag{14}$$

to get :

$$b = \frac{qQ}{4\pi\epsilon_0 mbv_0}\, \cot\left(\frac{\theta}{2}\right) \equiv \frac{qQ}{8\pi\epsilon_0 E_0}\, \cot\left(\frac{\theta}{2}\right) \quad \text{where} \quad E_0 = \frac{1}{2} m v_0^2 \tag{15}$$

2 Differential cross section

1) The nucleus has obviously the spherical symmetry and we have showed that the motion of the alpha particle is constrained in a plane. Hence, this plane will simply rotate about the x-axis depending on the initial conditions.

Let $\mathcal{J}$ be the incoming flux of alpha particles and $\dot{N}_s(d\Omega)$ the rate of scattered particles in the solide angle $d\Omega$. We have just shown that the scattering angle is uniquely determined by the impact parameter b. Thus, an alpha particle will be scattered in $d\Omega$ if and only if its impact parameter is between b and $b+db$. Due to the symmetry with respect to x-axis, this leads to a little ring of width db (see figure 2 in the questions sheet) whose area is nothing but $2\pi b db$.
We remind that a flux is a number of particles per unit time and per unit area. Then $\mathcal{J}$ can be seen as the number of particles crossing the little ring above mentionned per unit time divided by its area $2\pi b db$, meaning :

$$\mathcal{J} = \frac{\dot{N}_s(d\Omega)}{2\pi b db} \tag{16}$$

On the other hand, by definition of the cross section and since there only one target :

$$\dot{N}_s(d\Omega) = \mathcal{J} d\sigma \tag{17}$$

Combining (16) and (17), one finds

$$d\sigma = 2\pi b db \tag{18}$$

2) Here θ is the polar angle and let us denote φ the azimuthal angle. The solide angle expressed in spherical coordinates is well-known to be $d\Omega = \sin(\theta) d\theta d\varphi$. Nevertheless, because of the rotational symmetry about x-axis one can integral over φ which leads to :

$$d\Omega = 2\pi \sin(\theta) d\theta \tag{19}$$

Taking the ratio of (18) and (19) with absolute value since the b might decrease with θ but we want the differential cross section to be positive, we obtain :

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)} \left| \frac{db}{d\theta} \right| \tag{20}$$

Note the sine is always positive since $\theta \in [0, \pi]$.

3) We apply the general formula (20) to the relation scattering angle-impact parameter (15) for Rutherford scattering :

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin(\theta)} \frac{qQ}{8\pi\epsilon_0 E_0} \cot\left(\frac{\theta}{2}\right) \left| \frac{qQ}{8\pi\epsilon_0 E_0} \frac{1}{2} \frac{-1}{\sin^2\left(\frac{\theta}{2}\right)} \right| = \left(\frac{qQ}{4\pi\epsilon_0 E_0}\right)^2 \frac{\cos\left(\frac{\theta}{2}\right)}{2\sin(\theta)\sin^3\left(\frac{\theta}{2}\right)} \tag{21}$$

The nice trigonometric identity $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ leads to the famous Rutherford formula :

$$\frac{d\sigma}{d\Omega} = \left(\frac{qQ}{16\pi\epsilon_0 E_0 \sin^2\left(\frac{\theta}{2}\right)}\right)^2 \tag{22}$$