Problem 2062 (Princeton)

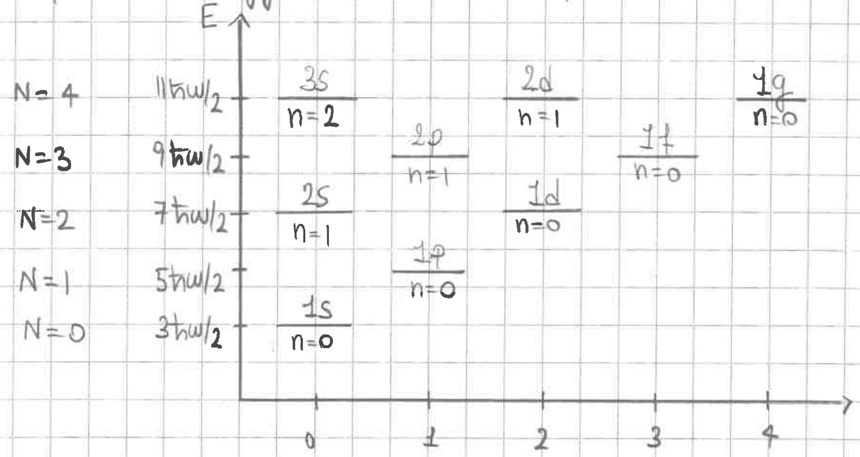
(a) $E = (2n+\ell+3/2)\hbar\omega = \left(N+\frac{3}{2}\right)\hbar\omega$, with $\hbar\omega = 44A^{-1/3}$ MeV

$n \in \mathbb{N}$. For any even N, we have $\ell = 0, \dots, N-2, N$, whereas for any odd N, we have $\ell = 1, \dots, N-2, N$. ℓ is even if N is such, and is odd if N is such.

Therefore, the level N will have parity $(-1)^N$.

The degeneracy at the level N is $d = \sum_{\ell=\dots,N-2,N} (2\ell+1) = \frac{(N+1)(N+2)}{2} \cdot 2$

The unperturbed energy levels are then,



Now we add the LS (spin-orbit) coupling to break the degeneracy.

$$\langle \ell s \rangle = \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right] =$$

$$= \begin{cases} \frac{\hbar^2}{2}\ell & , j = \ell + 1/2 \\ -\frac{\hbar^2}{2}(\ell+1) & , j = \ell - 1/2 \end{cases}$$

The potential becomes,

$$V_{tot}(r) = V_{HO}(r) + V_{\ell s}(r)\, \vec{L}\cdot\vec{S},$$

where $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momentum operators for a single nucleon and $V_{\ell s}$ is an arbitrary function of r (to be determined experimentally). The shift in energy for each level is given by,

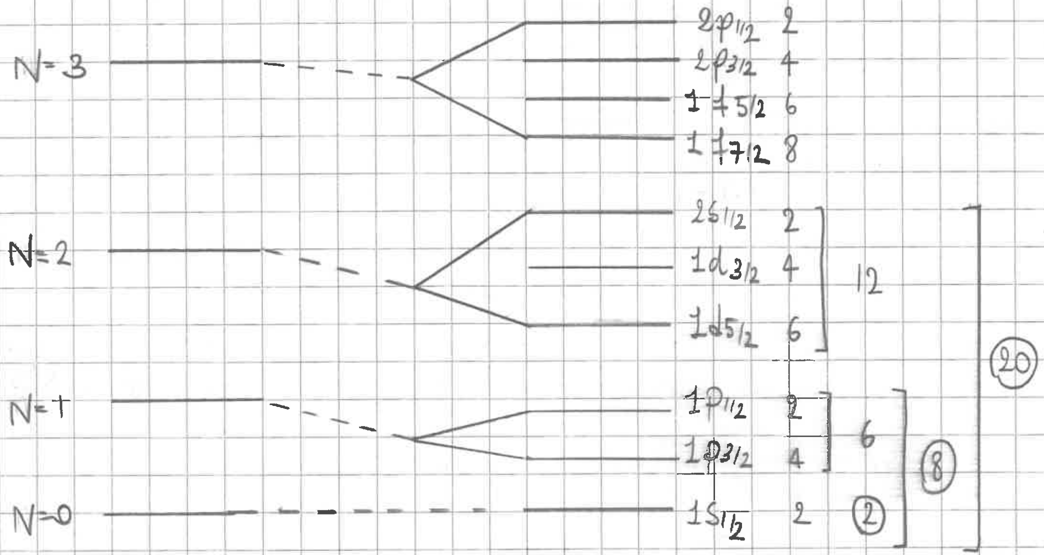
$$\Delta E = \langle \ell s \rangle \langle V_{\ell s}(r) \rangle = -\alpha \langle \ell s \rangle, \text{ with } \alpha > 0,$$

as determined empirically.

These can be computed as follows.

$l \backslash n$	0		1	
0	$N=0$	$1s$	$N=2$	$2s$
	$j=1/2$, $\Delta E = -\frac{l}{2}\hbar\alpha = 0$		$j=1/2$, $\Delta E = -l/2\,\hbar\alpha = 0$	
1	$N=1$	$1p$	$N=3$	$2p$
	$j=1/2$, $\Delta E = \alpha\hbar$	$j=3/2$, $\Delta E = \alpha\hbar/2$	$j=1/2$, $\Delta E = \alpha\hbar$	$j=3/2$, $\Delta E = \alpha\hbar/2$
2	$N=2$	$1d$		
	$j=3/2$, $\Delta E = \frac{3}{2}\alpha\hbar$	$j=5/2$, $\Delta E = -\alpha\hbar$		
3	$N=3$	$1f$		
	$j=5/2$, $\Delta E = 2\alpha\hbar$	$j=7/2$, $\Delta E = -\frac{3}{2}\alpha\hbar$		

Then, taking into account these corrections, the energy levels become,



(b) predict the ground state spins and parities of the following nuclei using the shell model:

$^{3}_{2}He$, $^{17}_{8}O$, $^{34}_{19}K$, $^{41}_{20}Ca$.

	p	n	$P = (-1)^{\sum_{i=\text{unpaired}} l_i}$	J^P
$^{3}_{2}He$	$(1s_{1/2})^2$	$(1s_{1/2})^1$	$(-1)^0 = 1$	$(1/2)^+$
$^{17}_{8}O$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$	$(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2 (1d_{5/2})^1$	$(-1)^2 = 1$	$(5/2)^+$
$^{34}_{19}K$	$(2s_{1/2})^1$	$(1d_{3/2})^1$	$(-1)^{0+2} = 1$	$(1)^+, (2)^+$
$^{41}_{20}Ca$	$(2s_{1/2})^2$	$(1f_{7/2})^1$	$(-1)^3 = -1$	$(7/2)^-$

(c) The selection rules for electric dipole transition are

$$\Delta J = J_f - J_i = 0, 1 \quad , \quad \Delta P = -1,$$

where J is the nuclear spin and P the nuclear parity. Since $\hbar\omega = 44A^{-1/3}$ MeV, $\hbar\omega > 5$ MeV for a nucleus (for $A = 300$, $\hbar\omega \simeq 6.57$ MeV). When N increases by 1, the energy level increases by more than 5eV, $\Delta E = \hbar\omega > 5$ MeV. Therefore, excited states with energy higher than the that of the ground state by less than 5 MeV must have the same N and parity than the ground state. Since electric dipole transitions require $\Delta P = -1$, such excited states cannot be connected to the ground state by an electric dipole transition.

However, in the spin-orbit corrections, the energy difference between states with different **N** can be less than 5 MeV, for heavy enough nuclei, so this transition is possible for them.

Exam, FK5024, 2018-01-04

6.(a) For an equation of state $p/c^2 = w\cdot\rho$, how does ρ depend on the scale factor a?

Solution: The continuity equation for matter in a homogeneous and isotropic universe reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 .$$

Since $p/c^2 = w\rho$, we obtain

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} .$$

This equation can be integrated with respect to time, giving the following solution.

$$\ln\left(\frac{\rho(t)}{\rho_0}\right) = -3(1+w)\ln\left(\frac{a(t)}{a_0}\right)$$

$$\rho(t) = \left[\rho_0 a_0^{3(1+w)}\right] a(t)^{-3(1+w)}$$

(b) In an exotic model with constituents X of the early Universe the energy density depends on the scale factor a as $\rho_X \propto a^{-3/2}$. What is the equation of state for the objects X?

Solution:

$$\rho_X \propto a^{-3/2} \Rightarrow \dot{\rho}_X \propto -\frac{3}{2}a^{-5/2}\dot{a} \Rightarrow \frac{\dot{\rho}_X}{\rho_X} = \frac{-\frac{3}{2}a^{-5/2}\dot{a}}{a^{-3/2}} = -\frac{3}{2}\frac{\dot{a}}{a}$$

$$a \propto \rho_X^{-2/3} \Rightarrow \dot{a} \propto -\frac{2}{3}\rho_X^{-5/3}\dot{\rho}_X \Rightarrow \frac{\dot{a}}{a} = \frac{-\frac{2}{3}\rho_X^{-5/3}\dot{\rho}_X}{\rho_X^{-2/3}} = -\frac{2}{3}\frac{\dot{\rho}_X}{\rho_X}$$

We now insert these expressions for the scale factor in the continuity equation.

$$\frac{\dot{\rho}_X}{\rho_X} = -3(1+w)\left(-\frac{2}{3}\frac{\dot{\rho}_X}{\rho_X}\right)$$

$$1 = -3(1+w)\left(-\frac{2}{3}\right) \Rightarrow 1 = 2(1+w) \Rightarrow w = -1/2 .$$

Since we know that $\dot{\rho}_X \neq 0$, it must be

$$\frac{11}{2} + \frac{p_X}{\rho_X c^2} = 0 \Rightarrow p_X/c^2 = -\frac{11}{2}\rho_X$$

You can also get this result using the formula from point (a),

$$\rho(t) \propto a^{-3(1+w)}, \quad \rho_X \propto a^{-3/2} \Rightarrow -3(1+w) = -3/2 \Rightarrow 1+w = \frac{1}{2} \Rightarrow w = -1/2 .$$

(c) How does the scale factor depend on time for this epoch when X dominates?

Solution: The acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_X + \frac{p_X}{c^2}\right) = -\frac{4\pi G}{3}\left(\rho_X - \frac{1}{2}\rho_X\right) = -\frac{4\pi G}{3}\rho_X$$

$$= -\frac{4\pi G}{3}\rho_X \propto -\frac{4\pi G}{3} a^{-3/2}$$

$$\ddot{a} \propto -a^{-1/2}.$$

Suppose that $a \propto \gamma t^{\beta}$. Then, $\ddot{a} \propto \gamma\beta(\beta-1)t^{\beta-2}$.

$$\gamma\beta(\beta-1)t^{\beta-2} \propto -\gamma^{-1/2} t^{-\beta/2}$$

We get,

$$\begin{cases} \gamma\beta(\beta-1) = -\gamma^{-1/2} \\ \beta - 2 = -\beta/2 \Rightarrow \frac{3}{2}\beta = 2 \Rightarrow \beta = \frac{4}{3} \end{cases}$$

$$\gamma\frac{4}{3}\left(\frac{4}{3}-1\right) = -\gamma^{-1/2} \Rightarrow \frac{4}{9}\gamma = -\gamma^{-1/2}$$

$$\gamma^{3/2} = -\frac{9}{4} \Rightarrow \gamma = \left(-\frac{9}{4}\right)^{2/3}$$

Therefore,

$$a(t) \propto \left(-\frac{9}{4}\right)^{2/3} t^{4/3} \propto t^{4/3}.$$

(8.2) $H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)$

$\cosh^{-1}(x) \simeq \ln(2x)$, for large x.

$\cosh^{-1}\left[\frac{1+x}{1-x}\right] \simeq 2\sqrt{x} + \frac{2}{3}x^{3/2}$, for small x.

$$\lim_{\Omega_0 \to 0}\left[\frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] \simeq$$

$$\simeq \lim_{\Omega_0 \to 0}\left[\frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \ln\left(2\,\frac{2-\Omega_0}{\Omega_0}\right)\right] = 1-0,$$

since Ω_0 goes to 0 faster than $\ln\left(2\,\frac{2-\Omega_0}{\Omega_0}\right)$ goes to infinity.

$$\lim_{\Omega_0 \to 1}\left[\frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] =$$

$$= \lim_{\Omega_0 \to 1}\left[\frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{1+(1-\Omega_0)}{1-(1-\Omega_0)}\right)\right]$$

We now define the variable $u = 1-\Omega_0$.

$$\lim_{\Omega_0 \to 1} u = 0.$$

$$\lim_{u \to 0}\left[\frac{1}{u} - \frac{1-u}{2u^{3/2}} \cosh^{-1}\left(\frac{1+u}{1-u}\right)\right] =$$

$$\lim_{u \to 0}\left[\frac{1}{u} - \frac{1-u}{2u^{3/2}}\left(2u^{1/2} + \frac{2}{3}u^{3/2}\right)\right] = \lim_{u \to 0}\left[\frac{1}{u} - \frac{1-u}{u} - \frac{(1-u)}{3}\right] =$$

$$= \lim_{u \to 0}\left[\cancel{\frac{1}{u}} - \cancel{\frac{1}{u}} + 1 - \frac{1}{3} + \frac{u}{3}\right] = \lim_{u \to 0}\left(\frac{2}{3} + \frac{u}{3}\right) = \frac{2}{3}$$