(9.1) $n_\gamma = n_\nu$, $E_\nu = m_\nu c^2$

$$\Omega_{rad} = \frac{\rho_{rad}}{\rho_c} \Rightarrow \rho_{rad} = \rho_c \Omega_{rad} \quad (\leftarrow \text{energy density})$$

$$\Rightarrow n_\gamma = \frac{\rho_{rad}}{E_\gamma} = \frac{\rho_c \Omega_{rad}}{E_\gamma} = \frac{\rho_c \Omega_{rad}}{3k_BT}$$

$$\Rightarrow n_\nu = \frac{\rho_c \Omega_{rad}}{3k_BT}$$

$$\cancel{\rho_\nu} = n_\nu E_\nu = \frac{\cancel{\rho_c}\Omega_{rad}}{3k_BT} m_\nu c^2$$

if the neutrinos contribute to a critical density: $\rho_c = \rho_\nu$

$$\Rightarrow m_\nu c^2 = \frac{3k_BT}{\Omega_{rad}} = \frac{3 \times 8,6.10^{-5} \times 2,725}{2,47.10^{-5}\ h^2}$$

$$= \frac{[eV].[K]^{-1}.[K]}{h^{-2}}$$

$$\Rightarrow \boxed{m_\nu c^2 = 28,5\ h^2\ eV}$$

→ Upper limit on the mass-energy:

$m_\nu c^2 \leqslant 10$ eV

$\Leftrightarrow$ $28,6\ h^2$ eV $\leqslant 10$ eV

$\Leftrightarrow$ $h \leqslant \sqrt{\frac{10}{28,6}}$ $\Leftrightarrow$ $\boxed{h \leqslant 0,6}$

→ More accurate calculation gives $E_\nu = 90\ h^2$ eV:

$\Rightarrow 90\ h^2 \leqslant 10$ eV $\Rightarrow \boxed{h \leqslant 0,3}$

→ Too small compared to measurements, so the electron neutrino is not a realistic DM candidate.

(9.2) $\rho_c = 2,78\ h^{-1}.10^{11} \frac{M_\odot}{(h^{-1} Mpc)^3}$

$\Omega_{DM} \simeq 0,25$

Number of black holes per Mpc^3: (eq° 9.8)

$h \simeq 0,6$

$$n_{BH} = \frac{\rho_{halo}}{m_{BH}} = \frac{\Omega_{DM}\ \rho_c}{m_{BH}}$$

$$= \frac{0,25 \times 2,78\ h^{-1}.10^{11}\ \cancel{[M_\odot]}}{10^{-10}\ \cancel{[M_\odot]}\ (h^{-1})^3 [Mpc]^3}$$

$\simeq 3.10^{20}\ Mpc^{-3}$

$\simeq 3.10^{20-18}\ pc^{-3}$

$\simeq 3.10^2\ pc^{-3}$

$Mpc = 10^6\ pc$
$(Mpc)^3 = 10^{6\times3}\ pc = 10^{18}\ pc$
$(Mpc)^{-3} = 10^{-18}\ pc$

$\Rightarrow \frac{1}{n_{BH}} \simeq \frac{1}{3.10^2}\ pc^3 = \boxed{3.10^{-3}\ pc^3}$ (solar syst radius: 10^{-4} pc)

So the closest BH is at $\sim 10^{-3}$ pc.

(10.1) CMB has $T \simeq 3$ K → how comes a microwave oven can heat food?

The waves from the oven don't produce a black-body radiation curve which should give you the temperature (Wien's law). And the food is not a black-body either.
The microwaves from the oven are generated with high intensity, which is not the case of the CMB. The photons from the CMB were generated at the beginning of the universe, and as the universe has expanded ever since, the photons have been redshifted and they have lost energy in the process.

(10.2) $\mathcal{E}_{rad} = \rho_{rad} c^2 = \alpha T^4$ (Stefan-Boltzmann)

→ Temperature of the universe at $t = 1s$?

Friedmann eq°: $H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$

$k = 0$ for our universe

Early times: radiation domination, $a(t) \propto t^{1/2}$

$\Rightarrow \dot a(t) = \frac{1}{2} t^{-1/2}$

Friedmann eq° becomes:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{rad}$$

$$\Rightarrow \rho_{rad} = \left(\frac{\frac{1}{2}t^{-1/2}}{t^{1/2}}\right)^2 \times \frac{3}{8\pi G}$$

$$\Rightarrow T = \left[\frac{1}{4t^2} \times \frac{3c^2}{8\pi G} \times \frac{1}{\alpha}\right]^{1/4}$$

$$T = \left(\frac{3 \times (2,998.10^8)^2}{4 \times 8 \times \pi \times 1^2 \times 6,672.10^{-11} \times 7,565.10^{-16}}\right)^{1/4}$$

$$= \left(\frac{\cancel{[m]}^2\ \cancel{[s]}^{-2}}{[J]\ \cancel{[m]}^{-3}\ [K]^{-4}\ \cancel{[s]}^2\ \cancel{[m]}^3\ [kg]^{-1}\ \cancel{[s]}^{-2}}\right)^{1/4}$$

$kg\ \cancel{m^2} \cancel{s^{-2}}$

$$\boxed{T \simeq 2.10^{10}\ K}$$

→ Mass density at that time?

$$\boxed{\text{mass density}} = \frac{\mathcal{E}_{rad}}{c^2} = \frac{J}{m^3} \times \frac{s^2}{m^2} = \frac{kg\ \cancel{m^2}\ \cancel{s^{-2}}}{m^3}\ \frac{\cancel{s^2}}{\cancel{m^2}} = \frac{kg}{m^3}$$

$$= \frac{\alpha T^4}{c^2}$$

$$= \frac{7,565.10^{-16} \times (2.10^{10})^4}{(2,998.10^8)^2}$$

$$\boxed{\simeq 2.10^9\ kg.m^{-3}}$$ water: $1000\ kg.m^{-3}$

~ a million times more than the mass density of water.

→ How old is the universe when its density = density of water?

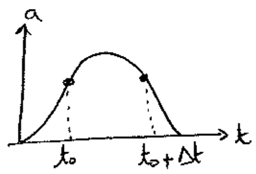
$$\rho_{rad} = \rho_{eau} = 10^3\ kg.m^{-3} = \left(\frac{1}{2t}\right)^2 \times \frac{3}{8\pi G}$$

$$\Rightarrow t = \sqrt{\frac{3}{10^3 \times 4 \times 8\pi G}}$$

$$\boxed{t \sim 1000\ s}.$$

(10.3) closed universe, $k = 1$, will recollapse.

eq° (10.7) : $T \propto \frac{1}{a}$ ↘



so when the universe will have again the same size as today, the temperature will be the same as today.

(10.6) We assume a critical density:

$$\rho = \rho_c \Rightarrow \Omega_m = 1,\ \Omega_\Lambda = 0,\ \Omega_k = 1$$

(eq 8.4:) $t_0 = \frac{2}{3H_0}$, $H_0^{-1} = 9{,}77\ h^{-1}.10^9\ yrs$

→ Radius of the last scattering surface?

$$R = c \times t_0 = c \times \frac{2}{3H_0} = c \times \frac{2}{3} H_0^{-1}$$

$$R = 3.10^8 \times \frac{2}{3} \times 9{,}77\ h^{-1}.10^9 \times 365 \times 24 \times 60^2\ m$$

$$R \simeq 6.10^{25}\ h^{-1}\ m$$

$$R \simeq \frac{6.10^{25}\ h^{-1}}{3{,}086.10^{16}.10^6}\ Mpc$$

$$\boxed{R \simeq 2000\, h^{-1}\ Mpc}$$

Here we probably underestimate the true value because we assumed a static universe.

→ Number of galaxies in the observable universe?

mass gal $= 10^{11}\ M_\odot$

eq (6.6) : $\rho_c(t_0) = 2{,}78\ h^{-1}.10^{11}\ \frac{M_\odot}{(h^{-1}Mpc)^3}$

$$N = \rho \times \frac{V_{universe}}{M_{gal}} = \rho_c \frac{V}{M_{gal}}$$

$$= 2{,}78\ h^{-1}.10^{11} \frac{\cancel{M_\odot}}{(h^{-1}\cancel{Mpc})^3} \times \frac{\frac{4}{3}\pi\ (2000\ h^{-1})\ \cancel{Mpc}^3}{10^{11}\ \cancel{M_\odot}}$$

$$\boxed{N \simeq 10^{11}\ \text{galaxies}}.$$

→ Nb of protons in the observable universe?

$\Omega = 1,\ \Omega_B \simeq 0{,}05$

$$M_{tot} = M_{gal} \times N_{gal} = 10^{11} \times 10^{11} = 10^{22}\ M_\odot$$

$$M_B \simeq \Omega_B M_{tot} \simeq 0{,}05.10^{22} \simeq 5.10^{20} M_\odot$$

$$n_p = \frac{M_B}{m_p}$$

$$m_p \simeq 1\ GeV = 1{,}7.10^{-27}\ kg = \frac{1{,}7.10^{-27}}{2.10^{30}}\ M_\odot \simeq 10^{-57}\ M_\odot$$

$$\Rightarrow n_p \simeq \frac{5.10^{20}}{10^{-57}} \Rightarrow \boxed{n_p \simeq 5.10^{77}}.$$

(11.3) $T_\odot \simeq 10^7\ K$

→ Age of the universe when $T = T_\odot$?

Radiation-domination (eq° 11.12) : $\left(\frac{1s}{t}\right)^{1/2} \simeq \frac{T}{1{,}3.10^{10}K}$

$$\Rightarrow t \simeq \left(\frac{1{,}3.10^{10}}{10^7}\right)^2 s$$

$$\boxed{t \simeq 2.10^6\ s}$$

$t < 1$ year, so the assumption of radiation domination is ok, since it is up to $\sim 10^4$ yrs.

→ CERN : 100 GeV :

$$T = \frac{E}{k_B} = \frac{100.10^9\ eV}{8{,}6.10^{-5}\ eV.K^{-1}}$$

$$\boxed{T \simeq 10^{15}\ K}$$

$$t \simeq \left(\frac{1{,}3.10^{10}}{10^{15}}\right)^2 \Rightarrow \boxed{t \simeq 10^{-10} s}.$$

(11.4) $\rho_c(t_0) = \frac{3H_0^2}{8\pi G} \simeq 10^4\ eV.cm^{-3}$

$\rho_\gamma(t_0) = \sigma_0 T_\gamma^4 = 0{,}25\ eV.cm^{-3}$ } see lecture notes

$$\Omega_{rad}(z_{dec}) = (1 + z_{dec})^4 \frac{\rho_\gamma(t_0)}{\rho_c(t_0)}$$

$$= (1100)^4.\frac{0{,}25}{10^4}$$

$$\simeq 0{,}03.$$

12.2 Alternate universe in which neutron half-life is 100s.

$$\text{eq}^\circ\ (12.9):\quad \frac{N_n}{N_p} \simeq \frac{1}{5}\exp\left(\frac{-340s\ \ln 2}{100s}\right)$$

$$\text{eq}^\circ\ (12.10):\quad Y_4 \equiv \frac{2N_n}{N_n+N_p} = \frac{2}{1+\frac{N_p}{N_n}}$$

$$\Rightarrow \boxed{Y_4 \simeq 0{,}04}.$$

12.3 We assume that the universe is charge neutral, so $n_p = n_e$.

$$n_B = n_n + n_p$$

$$\text{eq}^\circ\ (12.9):\quad \frac{n_n}{n_p} \simeq \frac{1}{7{,}3}$$

$$\Rightarrow n_B \simeq \frac{n_p}{7{,}3} + n_p \simeq \frac{n_e}{7{,}3} + n_e$$

$$\Rightarrow \frac{n_e}{n_B} \simeq \frac{1}{1+\frac{1}{7{,}3}}$$

$$\boxed{\frac{n_e}{n_B} \simeq 0{,}88}.$$

12.1 If the half-life of the neutron was of the order of a microsecond, then all neutrons would decay into protons before nucleosynthesis could happen (remember that nucleosynthesis happens at around 340s, see p.95 in Liddle).

Therefore, only protons and electrons would take part in the nucleosynthesis, and only hydrogen would form.

12.3 Decoupling happened when the Universe was 378000 years old, whereas nucleosynthesis happened when it was a few minutes old.

Hence, nucleosynthesis could be considered as a stronger test for the Big Bang model, then decoupling.

However, different people have different opinions about this. Remember that the CMB tells us that, at the time of decoupling, the Universe was extremely isotropic, and therefore the physics at decoupling is also of great importance for the Big Bang model.

(11.1) $\gamma = \sqrt[3]{\frac{11}{4}}$

$T_{ph} = \gamma T_\nu$

Since $T_\nu = \frac{T_{ph}}{\gamma}$ and $T \propto a^{-1}$, if we treat neutrino relativistically, then their density will be proportional to a^{-4}. Relatively to the density of photons, we then have

$$\Omega_\nu \propto a^{-4} \propto T_\nu^4 = \frac{T_{ph}^4}{\gamma^4} \propto a^{-4}\gamma^{-4} \propto \Omega_{ph}\gamma^{-4}$$

$$\Omega_\nu \simeq \left(\frac{4}{11}\right)^{4/3} \Omega_{ph}$$

Since there are three families of neutrinos, we have to multiply this by 3 and since they are fermions, an extra factor of $\frac{7}{8}$ appears, so

$$\Omega_\nu = 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3} \Omega_{ph} = 0.68\, \Omega_{ph} \quad (\text{eq. } 11.1)$$

(11.2) $n_\gamma \simeq 3.7 \cdot 10^8\, m^{-3}$ From the previous problem, we have

$$n_\nu = 0.68\, n_\gamma \simeq 2.5 \cdot 10^8\, m^{-3}$$

The average volume of a person of 70 Kg and a density of 985 $Kg\, m^{-3}$ is $V = 0.0711\, m^3 = 71\, \ell$.

Therefore: $N_\nu = n_\nu \cdot V \simeq 1.8 \cdot 10^7$

10.4 $n_e = 0.2\ m^{-3}$

$a \approx 10^{-6}$

$m_e = 0.511$ MeV

$$n(t_*) = \frac{n(t_0)}{a(t_*)^3} = 10^{18} n(t_0) = 2 \cdot 10^{17}\ m^{-3}$$

$$T(t_0) = 2.725\ K \Rightarrow T(t_*) = \frac{T(t_0)}{a(t_*)} = 10^6 \cdot 2.725\ K = 2.725 \cdot 10^6 K$$

$$E(t_*) = \left(8.62 \cdot 10^{-5} \frac{eV}{K}\right) 2.725\ K \simeq 2.35 \cdot 10^{-4}\ eV$$

$m_e \gg E(t_*) \Rightarrow$ electrons are non-relativistic

$\sigma_e = 6.7 \cdot 10^{-29} m^2$

$d \simeq \frac{1}{n_e \sigma_e}$

$c = 3 \cdot 10^8 ms^{-1}$

age ~~of~~ Universe $\simeq$ 10 000 yrs $= t_{un}$

$$d = \frac{1}{2 \cdot 10^7 m^{-3} \cdot 6.7 \cdot 10^{-29} m^2} \simeq 7.5 \cdot 10^{10} m$$

The time taken by photons to travel a space d is

$$t_{int} = d/c = 250 s$$

$t_{int} \ll t_{un}$, which means that there have been many interactions until this time