(6.1) The deceleration parameter is defined by equation (6.14):

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{a(t_0)\,\ddot{a}(t_0)}{\dot{a}^2(t_0)}$$

Acceleration equation (3.18): $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$

Definition of the critical density: special value of the density which would be required in order to make the geometry of the universe flat (k=0).

$$\rho_c(t) = \frac{3H^2}{8\pi G} \qquad \left(H = \frac{\dot{a}}{a}\right)$$

→ Show that a radiation-dominated universe has $q_0 = \Omega_0$?

$\Omega_0 = \Omega(t_0)$ = density parameter today

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)}$$

We have seen in exercise (5.2) that the equation of state for radiation is: $p = \frac{1}{3}\rho c^2$

→ The acceleration equation becomes:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho + \frac{3}{c^2}\,\frac{1}{3}\rho c^2\right) = -\frac{8\pi G}{3}\rho$$

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = \frac{8\pi G}{3}\rho_0\frac{1}{H_0^2} = \frac{\rho_0}{\rho_c(t_0)}$$

$$\Rightarrow \boxed{q_0 = \Omega_0}.$$

(6.2) Condition on the equation of state in order to have $q_0 < 0$? (ie. acceleration)

$$q_0 < 0 \iff -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} < 0$$

$$\iff \frac{4\pi G}{3}\left(\rho(t_0) + \frac{3}{c^2}p(t_0)\right)\frac{1}{H_0^2} < 0$$

$$\iff \rho(t_0) + \frac{3}{c^2}p(t_0) < 0$$

$$\iff \boxed{p < -\frac{c^2\rho}{3}}$$

Since $\rho > 0$ and $c > 0$, this means $p < 0$.

So that means that the accelerated expansion of the universe is possible only if we have a component with a negative pressure (dark energy).

(7.1) • Radiation: see (5.3.2): $\rho \propto \frac{1}{a^4}$

• Non-relativistic matter: see (5.3.1): $\rho \propto \frac{1}{a^3}$

• Cosmological constant: ρ = constant by definition (otherwise you can use fluid equation for Λ)

• Negative curvature: $k=-1$: $\rho \propto \frac{1}{a^2}$

(Friedmann eq°: $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$)

At early times (ie. for small a), radiation dominates. Then at late times, Λ dominates.

(7.2)
$$\begin{cases} \text{Friedmann: } H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ \text{Acceleration: } \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \cancel{\frac{3p}{c^2}}\right) + \frac{\Lambda}{3} \end{cases}$$

We assume a pressureless universe: $p = 0$.

We have a static universe if the scale factor is constant $\Rightarrow \dot{a} = \ddot{a} = 0$.

$$\Rightarrow \begin{cases} \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} = 0 \\ -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} = 0 \end{cases}$$

$\Rightarrow \Lambda = 4\pi G\rho > 0$ → positive vacuum energy.

$\Rightarrow \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{4\pi G}{3}\rho = 0$

$\Rightarrow k = a^2\, 4\pi G\rho > 0$ → closed universe.

Why is this solution unstable?

If the radius of the universe is a little bit larger, the gravity (attractive) will decrease (since it will be acting over larger distances) but the cosmological constant (acting against gravity) will remain constant and unchanged. This way the Λ will win and the universe will expand forever, more and more rapidly.

On the contrary if we take a universe having a slightly smaller radius, gravity wins over Λ and the universe will contract more and more rapidly.

(7.3) Pressureless universe with Λ :

$$\begin{cases} \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} & \text{(acceleration eq°)} \\ q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} & \text{(deceleration parameter)} \\ \rho_c(t) = \frac{3H^2}{8\pi G} = \frac{\rho(t)}{\Omega(t)} & \text{(critical density)} \end{cases}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_c(t)\Omega(t) + \frac{\Lambda(t)}{3}$$

$$= -\frac{4\pi G}{3}\frac{3H^2}{8\pi G}\Omega(t) + \frac{\Lambda(t)}{3}$$

$$= -\frac{H^2}{2}\Omega(t) + \frac{\Lambda(t)}{3}$$

$$\Rightarrow q_0 = -\frac{1}{H_0^2}\left(-\frac{H_0^2}{2}\Omega(t_0) + \frac{\Lambda(t_0)}{3}\right)$$

$$= \frac{\Omega_0}{2} - \frac{\Lambda(t_0)}{3H_0^2}$$

We know that by definition $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$

and $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \rho_\Lambda \frac{8\pi G}{3H^2} = \frac{\Lambda}{3H^2}$

$$\Rightarrow \boxed{q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda(t_0)}.$$

(8.1) (Definition of a supernova? Discussion about stellar evolution).

$$\left.\frac{U^{235}}{U^{238}}\right)_{initial} \simeq 1,65 \qquad \left.\frac{U^{235}}{U^{238}}\right)_{final} \simeq 0,0072$$

decay rates: $\begin{cases} \lambda(U^{235}) = 0,97.10^{-9}\ yr^{-1} \\ \lambda(U^{238}) = 0,15.10^{-9}\ yr^{-1} \end{cases}$

Decay law : $U(t) = U(0)\,e^{-\lambda t}$

→ Age of the galaxy?

$$\begin{cases} U_{235}(t) = U_{235}(0)\,e^{-\lambda_{235} t} \\ U_{238}(t) = U_{238}(0)\,e^{-\lambda_{238} t} \end{cases}$$

$$\Rightarrow \frac{U_{235}(t)}{U_{238}(t)} = \frac{U_{235}(0)}{U_{238}(0)}\,e^{-\lambda_{235}t + \lambda_{238}t}$$

$$\Rightarrow 0,0072 \simeq 1,65\,e^{t(0,15-0,97).10^{-9}}$$

$$\Rightarrow \boxed{t \simeq 6,6 \text{ billion years}}.$$

We now assume that actually $t \simeq 7,6.10^9$ yrs, and a critical-density universe ($\rho = \rho_c$)

→ Upper limit on the value of h?
(h is the uncertainty on H_0).

If the universe has a critical density, see equation (5.15): $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$

$$\Rightarrow H \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \qquad \Rightarrow H_0 = \frac{2}{3t_0}.$$

$$t_0 = \frac{2}{3}H_0^{-1}$$

eq°(8.2): $H_0^{-1} = 9,77\ h^{-1}.10^9$ yrs

$$\Rightarrow 7,6.10^9 = \frac{2}{3}\times 9,77\ h^{-1}.10^9$$

$$\Rightarrow \boxed{h = 0,85}.$$

(8.3) Why introducing a positive Λ will increase the age of the universe?

$\Lambda > 0$ means acceleration of the expansion. So at early times the expansion was not as fast. So it took longer to arrive at what it is today.

7.2 $\quad H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \cancel{\frac{3p}{c^2}}\right) + \frac{\Lambda}{3}$$

In order to have $\dot a = \ddot a = 0$ (i.e., a static universe) we saw that

$$\Lambda_0 = 4\pi G\rho > 0 \quad , \quad k_0 = a^2 4\pi G\rho > 0 ,$$

so $k = a^2\Lambda$. Let's perturb a little these values

$$\Lambda = \Lambda_0 + \delta\Lambda \quad , \quad k = k_0 + \delta k ;$$

The Friedmann and acceleration equations are

$$\begin{cases} H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} = \cancel{\left[\frac{8\pi G}{3}\rho - \frac{k_0}{a^2} + \frac{\Lambda_0}{3}\right]} - \frac{\delta k}{a^2} + \frac{\delta\Lambda}{3} = -\frac{\delta k}{a^2} + \frac{\delta\Lambda}{3} \\ \frac{\ddot a}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} = \cancel{\left[-\frac{4\pi G}{3}\rho + \frac{\Lambda_0}{3}\right]} + \frac{\delta\Lambda}{3} = \frac{\delta\Lambda}{3} \end{cases}$$

$$\frac{\ddot a}{a} = \frac{\delta\Lambda}{3} \implies \frac{d^2 a}{dt^2} - \frac{\delta\Lambda}{3}a = 0$$

If $\delta\Lambda > 0$ then the solution with $a(0) = a_0$ is $a(t) = a_0\cosh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)$.
If $\delta\Lambda < 0$, the equation becomes $\frac{d^2 a}{dt^2} + \frac{|\delta\Lambda|}{3}a = 0$ and the solution with $a(0) = a_0$ is $a(t) = a_0\cos\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)$.
Let's plug this into the Friedmann equation.

$\boxed{\delta\Lambda > 0}$

$$\left[\frac{\sqrt{\frac{\delta\Lambda}{3}}\sinh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)}{\cosh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)}\right]^2 = -\frac{\delta k}{a_0^2\cosh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)^2} + \frac{\delta\Lambda}{3}$$

$$\frac{\delta\Lambda}{3}\left[\sinh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)^2 - \cosh\left(\sqrt{\frac{\delta\Lambda}{3}}t\right)^2\right] = -\frac{\delta k}{a_0^2}$$

$$-\frac{\delta\Lambda}{3} = -\frac{\delta k}{a_0^2} \implies \delta k = \frac{a_0^2\delta\Lambda}{3} > 0.$$

So we have

$a(t) = a_0\cosh\left[\sqrt{\frac{\delta\Lambda}{3}}t\right]$,

i.e., even small $\delta\Lambda, \delta k > 0$ make $a(t)$ to grow indefinitely.

$\boxed{\delta\Lambda < 0}$

$$\left[\frac{-\sqrt{\frac{|\delta\Lambda|}{3}}\sin\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)}{\cos\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)}\right]^2 = -\frac{\delta k}{a_0^2\cos\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)} - \frac{|\delta\Lambda|}{3}$$

$$\frac{|\delta\Lambda|}{3}\left[\sin\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)^2 + \cos\left(\sqrt{\frac{|\delta\Lambda|}{3}}t\right)^2\right] = -\frac{\delta k}{a_0^2}$$

$$\frac{|\delta\Lambda|}{3} = -\frac{\delta k}{a_0^2} \implies \delta k = -\frac{a_0^2|\delta\Lambda|}{3} < 0$$

So we have

$a(t) = a_0\cos\left[\sqrt{\frac{|\delta\Lambda|}{3}}t\right]$,

i.e., also in this case the perturbation will not decrease in amplitude when t increases.