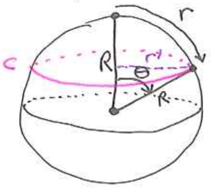
(4.1) We consider the surface of a 2D sphere of radius R:



→ What is the circumference of the circle c (on the surface of the sphere)?

$$\left.\begin{aligned} c &= 2\pi r' \\ \sin\theta &= \frac{r'}{R} \end{aligned}\right\} \Rightarrow c = 2\pi R \sin\theta$$

By definition of angles in radians, we have:

$$r = \theta R$$

Which gives:

$$\boxed{c = 2\pi r \frac{\sin\theta}{\theta}} \quad \text{or} \quad \boxed{c = 2\pi R \sin\left(\frac{r}{R}\right)}$$

→ Now we consider small θ (i.e. $r \ll R$):

$\Rightarrow \sin\theta \sim \theta$

$$\Rightarrow \boxed{c} = 2\pi r \frac{\sin\theta}{\theta} \sim 2\pi r \frac{\cancel{\theta}}{\cancel{\theta}} = \boxed{2\pi r}$$

which is the normal flat relation for circumference (it means that for small angles, the surface of the sphere can be approximated to be flat).

→ Situation when the circle c is at the equator:

$\theta = \frac{\pi}{2} \Rightarrow \sin\theta = 1$

$$\Rightarrow \boxed{c} = 2\cancel{\pi} r \frac{1}{\frac{\cancel{\pi}}{2}} = \boxed{4r}$$

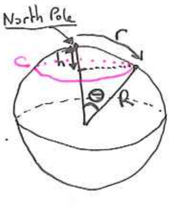
(4.2) We consider the same geometry as in 4.1 (ie. 2D analogy to the real Universe).

We suppose that galaxies are distributed evenly in there, with n = nb density per unit area.

→ Total number of galaxies N inside a radius r?

$N = n \times A$

with A being the surface area on the sphere from North Pole to the circle c.



$A = 2\pi R h$

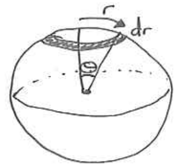
and we have: $h = R - R\cos\theta$

$\Rightarrow N = n\, 2\pi R^2 (1 - \cos\theta)$

$$\boxed{N = 2\pi n R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right)} \quad (r = \theta R).$$

→ Now we consider again $r \ll R$, $\theta \sim 0$:

We consider a circular strip of width dr, at radius r.



The area of the strip is:

$$A_s = c \times dr$$
$$= 2\pi R \sin\left(\frac{r}{R}\right) dr$$

For $r \ll R$, $\theta \sim 0 \rightarrow \sin\theta \sim \theta$

So: $A_s \sim 2\pi \cancel{R} \frac{r}{\cancel{R}}\, dr$

$$A_s \sim 2\pi r\, dr$$

Then the area from North Pole to c is:

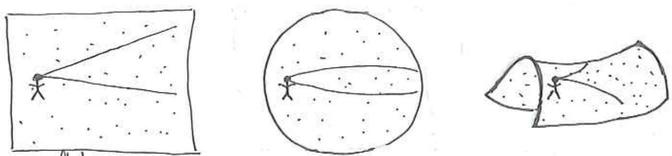
$$A = \int_0^r A_s = \int_0^r 2\pi r\, dr = 2\pi \left[\frac{r^2}{2}\right]_0^r = \pi r^2$$

$$\Rightarrow \boxed{N = n\pi r^2}$$

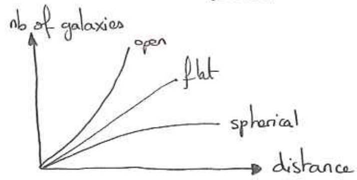
→ Do we see more galaxies in a spherical or a flat Universe?

We just calculated that the numbers of galaxies are:

- for a spherical Universe: $N = 2\pi n R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right)$
- for a flat Universe: $N = n\pi r^2$



flat Universe   Spherical Universe (closed)   (Open Universe)



Fewer galaxies are seen in the spherical geometry.

(5.2) An ideal gas has pressure $p = \frac{1}{3} n \langle \vec{v} . \vec{p} \rangle$

nb density — average over direction of particle motions — momentum

Equation (2.4): $E_{tot}^2 = m^2c^4 + p^2c^2$

$\Leftrightarrow p^2c^2 = E^2 - m^2c^4$

$\Leftrightarrow p^2 = \frac{E^2}{c^2} - m^2c^2$

for photons : $m = 0$ (no rest mass)

$\Rightarrow p^2 = \frac{E^2}{c^2} \Rightarrow p = \frac{E}{c}$

$\Rightarrow p = \frac{1}{3} n \langle \vec{V} . \frac{E}{c} \rangle$

for photons : $V = c$

$\Rightarrow \boxed{p = \frac{1}{3} n \langle E \rangle}$

Hence demonstrate the equation of state for radiation:

$\langle E \rangle = mc^2$

$\Rightarrow p = \frac{1}{3} n m c^2$

and $nm = \rho$ (density) because $\begin{cases} \rho = \frac{[m]}{[V]} \\ n = \frac{nb}{[V]} \end{cases}$

$\Rightarrow \boxed{p = \frac{1}{3} \rho c^2}$

(5.3) We consider now a more general equation of state : $p = (\gamma - 1) \rho c^2$ with $\gamma =$ cste, $0 < \gamma < 2$.

We assume $k = 0$.

- Solution for $\rho(a)$?

Fluid equation (5.2) :

$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) = 0$ (then proceed like in 5.3.1)

$\Leftrightarrow \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + (\gamma - 1) \rho) = 0$

$\Leftrightarrow \dot{\rho} + 3\gamma \frac{\dot{a}}{a} \rho = 0$

We can come with the idea that:

$\frac{1}{a^3} \frac{d}{dt} (\rho a^3) = \frac{1}{a^3} (\dot{\rho} a^3 + \rho 3 a^2 \dot{a}) = \dot{\rho} + 3 \rho \frac{\dot{a}}{a}$

And then we deduce :

$\frac{1}{a^{+3\gamma}} \frac{d}{dt} \left(\rho a^{+3\gamma} \right) = \frac{1}{a^{+3\gamma}} \left(\dot{\rho} a^{+3\gamma} + \rho (+3\gamma) a^{+3\gamma - 1} \dot{a} \right)$

$= \dot{\rho} + 3\gamma \rho \frac{a^{+3\gamma - 1} \dot{a}}{a^{+3\gamma}}$

$= \dot{\rho} + 3\gamma \rho \frac{\dot{a}}{a}$

Which implies:

$\Rightarrow \frac{1}{a^{+3\gamma}} \frac{d}{dt} (\rho a^{+3\gamma}) = 0 \Rightarrow \boxed{\rho(a) \propto a^{-3\gamma}}$.

- Solution for $a(t)$?

$\rho \propto a^{-3\gamma} \Rightarrow \rho = \rho_0 a^{-3\gamma}$

Friedmann equation (5.1) :

$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \cancel{\frac{k}{a^2}}$ $(k = 0)$

$\Rightarrow \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3\gamma}$

$\Rightarrow \dot{a}^2 = \frac{8\pi G \rho_0}{3} a^{-3\gamma + 2}$

$\Rightarrow \dot{a} = \sqrt{\frac{8\pi G \rho_0}{3}} a^{-\frac{3}{2}\gamma + 1}$

A good guess in cosmology is : $a \propto t^q$

→ if we consider so :

$a \propto t^q \Rightarrow \dot{a} \propto q t^{q-1}$

→ if we substitute in the equation:

$q t^{q-1} \propto \sqrt{\frac{8\pi G \rho_0}{3}} t^{q(-\frac{3}{2}\gamma + 1)}$

$\Rightarrow q - 1 = q \left(-\frac{3}{2}\gamma + 1 \right)$

$\Rightarrow q = \frac{2}{3\gamma}$

$\Rightarrow \boxed{a(t) \propto t^{\frac{2}{3\gamma}}}$.

- Solution for $\rho(t)$?

$\rho = \rho_0 a^{-3\gamma} \Rightarrow \rho \propto \rho_0 t^{\frac{2}{3\gamma}(-3\gamma)}$

$\Rightarrow \boxed{\rho \propto \rho_0 t^{-2}}$.

- Solution if $p = -\rho c^2$?

Since $p = (\gamma - 1) \rho c^2$, that means that $\gamma = 0$.

Then $\rho \propto a^{-3\gamma}$ becomes $\rho = \text{cste} = \rho_0$

$\Rightarrow \left(\frac{\dot{a}}{a} \right)^2 = \text{cste} = \frac{8\pi G \rho_0}{3}$

$\Rightarrow \dot{a} = \sqrt{\frac{8\pi G \rho_0}{3}} a$

$\Rightarrow \boxed{a \propto \exp\left(\sqrt{\frac{8\pi G \rho_0}{3}} t \right)}$.

(5.4) We want to have $\rho \propto \frac{k}{a^2} = k a^{-2}$

According to (5.3) $\rho \propto a^{-3\gamma}$

$\Rightarrow -3\gamma = -2 \Rightarrow \boxed{\gamma = \frac{2}{3}}$.

Now with $\gamma = \frac{2}{3}$ and assuming $k = -1$, what is the solution $a(t)$ to the Friedmann eq° ?

Friedmann eq° : $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{1}{a^2}$ (if $k=-1$)

$$\Rightarrow \left(\frac{\dot a}{a}\right)^2 = -\frac{8\pi G}{3}\frac{1}{a^2} + \frac{1}{a^2}$$

$$\Rightarrow \dot a^2 = 1 - \frac{8\pi G}{3}$$

$$\Rightarrow \dot a = \sqrt{1 - \frac{8\pi G}{3}}$$

$$\dot a = \text{cste}$$

$$\Rightarrow \boxed{a(t) \propto t}.$$

(5.5) We consider a universe with $k=1$, which contains only matter ($p=0 \Rightarrow \rho = \frac{\rho_0}{a^3}$)

$$a(\theta) = \frac{4\pi G\rho_0}{3k}(1-\cos\theta) \;;\; t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta)$$

Chain rule derivative : $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$$\dot a = \frac{da}{dt} = \frac{da}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\begin{cases} \frac{da}{d\theta} = \frac{4\pi G\rho_0}{3k}\sin\theta \\ \frac{dt}{d\theta} = \frac{4\pi G\rho_0}{3k^{3/2}}(1-\cos\theta) \end{cases}$$

$$\Rightarrow \dot a = \frac{4\pi G\rho_0}{3k}\sin\theta \; \frac{3k^{3/2}}{4\pi G\rho_0(1-\cos\theta)}$$

$$\dot a = k^{1/2}\frac{\sin\theta}{1-\cos\theta}$$

→ We inject a and $\dot a$ into Friedmann equation:

$$\left(\frac{\dot a}{a}\right)^2 - \frac{8\pi G}{3}\rho + \frac{k}{a^2}$$

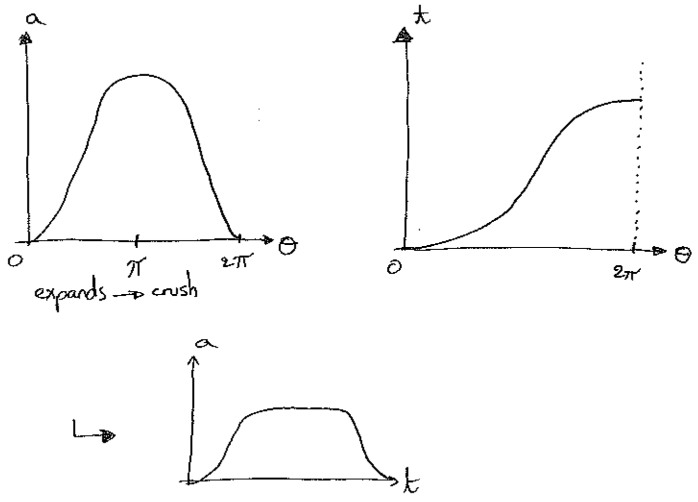
$$= k\frac{\sin^2\theta}{(1-\cos\theta)^2}\times\frac{9k^2}{(4\pi G\rho_0)^2(1-\cos\theta)^2} - \frac{8\pi G\rho_0}{3}\times\frac{3\times 9k^3}{(4\pi G\rho_0)^3(1-\cos\theta)^3} + k\frac{9k^2}{(4\pi G\rho_0)^2(1-\cos\theta)^2}$$

$$= \frac{9k^3}{(4\pi G\rho_0)^2}\left[\frac{\sin^2\theta}{(1-\cos\theta)^4} - \frac{2}{(1-\cos\theta)^3} + \frac{1}{(1-\cos\theta)^2}\right]$$

$$= \frac{9k^3}{(4\pi G\rho_0)^2}\left[\frac{\sin^2\theta - 2 + \cancel{2\cos\theta} + 1 + \cos^2\theta - \cancel{2\cos\theta}}{(1-\cos\theta)^4}\right]$$

$$= 0$$

→ So $a(\theta)$ is solution of Friedmann equation.



(5.6) We consider a universe containing only matter ($p = 0 \rightarrow \rho = \frac{\rho_0}{a^3}$) with $k=-1$.

Here we consider that the final term of the Friedmann eq° dominates over the density term :

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k^2}{a^2} \longrightarrow \left(\frac{\dot a}{a}\right)^2 \sim -\frac{k}{a^2} = \frac{1}{a^2}$$

$$\Rightarrow \dot a = 1 \quad \Rightarrow \boxed{a \propto t}$$

$$\rho = \frac{\rho_0}{a^3} \quad \Rightarrow \boxed{\rho \propto t^{-3}}.$$

which means that ρ will keep decreasing with time, so the domination by curvature will continue forever (as a increases with time). It is a stable situation.

(5.1) Discussion : is the total energy of the universe conserved as it expands ?

→ The first answer that comes to mind should be "of course ! Energy is always conserved."

But are we sure of that ?

We have seen that the Universe is in an accelerating expansion, caused by an energy that is filling the (expanding) space.

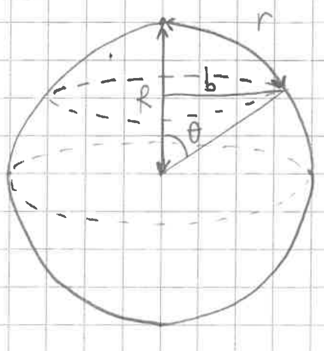
So, as space expands, so does the amount of energy ... ?

↓

True, energy is not conserved in GR: it changes because space-time evolves.

(exple: if the space in which particles are moving is changing, the total energy of those particles isn't conserved, cf. with the redshift : photons are losing energy as space expands, leaving us with the same nb of photons, but a lower energy per photon).

(4.1)



The arc length r is given by

$$r = R\theta, \qquad (1)$$

whereas b is equal to

$$b = R\sin(\theta).$$

Being $R = \frac{r}{\theta}$ from the first formula, we get

$$b = \frac{\sin(\theta)}{\theta} r.$$

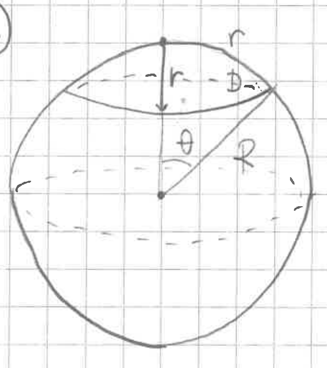
Therefore the circumference of such a circle is given by

$$C = 2\pi b = 2\pi \frac{\sin(\theta)}{\theta} r.$$

We can also isolate θ rather than R in (1) and obtain $\theta = \frac{r}{R}$, which gives us

$$C = 2\pi b = 2\pi R \sin\left(\frac{r}{R}\right).$$

(4.2)



The number of galaxies inside a radius r on the 2-sphere is equal to the uniform density n times the area A of this "circle on the 2-sphere".

$$A(r) = \int_D dA = R^2 \int_0^{2\pi} d\varphi \int_0^{\theta} \sin(\theta) d\theta,$$

since the surface element on the sphere is $R^2 \sin(\theta) d\theta d\varphi$.

Also, we know that $\theta = \frac{r}{R}$, so

$$A(r) = \int_0^{2\pi} d\varphi \int_0^{r/R} R^2 \sin(\theta) d\theta = 2\pi R^2 \left(-\cos(\theta)\right)\Big|_0^{r/R} = 2\pi R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right),$$

The number of galaxies inside a radius r is then

$$N(r) = 2\pi n R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right).$$

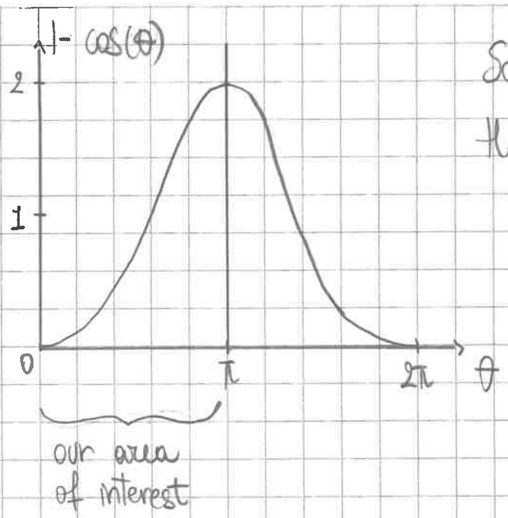
If $r \ll R$, that is in the limit $R \to \infty$ corresponding to "flattening" the sphere into a plane, we get $\cos\left(\frac{r}{R}\right) = 1 - \frac{1}{2}\left(\frac{r}{R}\right)^2 + O(r^4)$. It follows

$$N(r)\Big|_{R\to\infty} \simeq 2\pi n R^2 \left(1 - 1 + \frac{1}{2}\frac{r^2}{R^2}\right) = \pi r^2 \cdot n = N_f(r).$$

which is the well-known result in flat-space. Let's compare the two results.

$$N(r) - N_f(r) = 2\pi n R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right) - \pi n r^2 =$$

$$= 2\pi n \, r^2/\theta^2 \left(1 - \cos(\theta)\right) - \pi n r^2 = \frac{2\pi n r^2}{\theta^2}\left[1 - \cos(\theta) - \frac{\theta^2}{2}\right]$$



So the sign of $N(r)-N_f(r)$ depends on the sign of $1-\cos(\theta)-\frac{\theta^2}{2}$.

$$\frac{d}{d\theta}\left(1-\cos(\theta)-\frac{\theta^2}{2}\right) = \sin(\theta)-\theta$$

$$\frac{d}{d\theta}\left(\sin(\theta)-\theta\right) = \cos(\theta)-1 = -(1-\cos(\theta)) \leq 0$$

Since $\left.(\sin(\theta)-\theta)\right|_{\theta=0} = 0$ and its derivative is negative, $\sin(\theta)-\theta\leq 0$ and the equality holds only at $\theta=0$. This means that $1-\cos(\theta)-\frac{\theta^2}{2}\leq 0$ for the same reasons.

Therefore

$$N(r)-N_f(r) = \frac{2\pi n r^2}{\theta^2}\left[1-\cos(\theta)-\frac{\theta^2}{2}\right] \leq 0 ,$$

with the equality only at $\theta=0$, i.e. zero galaxies. This means that, for

$\theta\in(0,\pi]$, $N_f(r) > N(r)$,

That is, we see fewer galaxies out of the same radius, if the Universe is spherical rather than flat.

5.1 In Section 3.1, the Friedmann equation is derived starting from conservation of energy.

$$U = T + V \quad (\text{eq } 3.6) \Longrightarrow U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G \rho a^2 x^2 m \quad (\text{eq } 3.9)$$

Now multiply by $\frac{2a}{mx^2}$,

$$a\frac{2U}{mx^2} = \dot{a}^2 a - \frac{8\pi G}{3}\rho a^2 a$$

$$\dot{a}^2 a - \frac{8\pi G}{3}\rho a^3 - \frac{2Ua}{mx^2} = 0$$

$$\dot{a}^2 a - \frac{8\pi G}{3}\rho a^3 + kc^2 a = 0, \quad \text{with } kc^2 = -\frac{2U}{mx^2}$$

$$\frac{8\pi G}{3}\rho a^3 = \dot{a}^2 a + kc^2 a, \quad \frac{8\pi G}{3}\rho a^3 - \left[\dot{a}^2 a + kc^2 a\right] = 0$$

So the energy associated with the matter (any type of matter, radiation, cosmological constant) is perfectly balanced by the energy of the gravitational interaction in the term $\dot{a}^2 a + ka$. This means that the energy of matter alone is not conserved.

For a rigorous treatment in general relativity, one needs to introduce the Landau-Lipshitz pseudotensor.

(5.1) The total energy of the Universe is conserved, as the Universe expands. However, to achieve this conservation we need to take into account the contribution of the gravitational interaction. To do this, we need to take into account general relativity. If we take the redshift of photons as an example, we notice that they lose energy during their trip to us. The energy loss is due to work done by the photons against the general relativistic gravitational potential, so energy is conserved when we take into account general relativity. This phenomenon is called gravitational redshift.

(5.2) $E^2 = m^2c^4 + p^2c^2$ A photon has $m=0$, therefore

$$E^2 = p^2c^2 \Longrightarrow E = pc = |\vec{p}|c$$

The ideal gas equation of state reads

$$p = \frac{1}{3} n \langle \vec{v} \cdot \vec{p} \rangle$$

$\vec{v}$ and $\vec{p}$ are parallel, therefore $\vec{v} \cdot \vec{p} = |\vec{v}| \cdot |\vec{p}| = c \cdot \frac{E}{c} = E$, which implies

$$p = \frac{1}{3} n \langle E \rangle, \text{ with } \langle E \rangle \text{ mean energy of one photon.}$$

Now we can define $\epsilon := n\langle E \rangle$ as the mean energy density of the ideal gas of photon, and get $p = \frac{\epsilon}{3}$. Dimensionally we know that $\epsilon = \rho c^2$, with ρ mass density, so we can write

$$p = \frac{1}{3}\rho c^2.$$

(5.3) $p = (\gamma - 1)\rho c^2$ Start by solving the conservation equation for matter.

$0 < \gamma < 2$

$k = 0$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{(\gamma-1)\rho c^2}{c^2}\right) = 0$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\cancel{\rho} + \gamma\rho - \cancel{\rho}\right) = 0$$

$$\dot{\rho} + 3\gamma\frac{\dot{a}}{a}\rho = 0 \Longrightarrow \frac{\dot{\rho}}{\rho} = -3\gamma\frac{\dot{a}}{a}$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -3\gamma \ln\left(\frac{a}{a_0}\right) = \ln\left[\left(\frac{a_0}{a}\right)^{3\gamma}\right]$$

$$\rho(t) = \rho_0\left(\frac{a_0}{a}\right)^{3\gamma}$$

Usually a_0 is the value of the scale factor today, so $a_0 = 1$.

$$\rho(t) = \frac{\rho_0}{a(t)^{3\gamma}}$$

Now we can plug $\rho(t)$ into the Friedmann equation and find $a(t)$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad , \text{ with } k=0.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^{3\gamma}}$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0\, a^{2-3\gamma}$$

Let's suppose that $a(t) = c\,t^{\beta}$. We have $\dot{a} = c\beta t^{\beta-1}$, so

$$c^2\beta^2 t^{2\beta-2} = \frac{8\pi G}{3}\rho_0 (ct)^{2\beta-3\gamma\beta} = \left[\frac{8\pi G}{3}\rho_0\, c^{2\beta-3\gamma\beta}\right] t^{2\beta-3\gamma\beta}$$

$$\begin{cases} \cancel{2\beta}-2 = \cancel{2\beta}-3\gamma\beta \\ c^2\beta^2 = \frac{8\pi G}{3}\rho_0\, c^{2\beta-3\gamma\beta} \end{cases}$$

The first equation gives us $\beta = \frac{2}{3\gamma}$. We can plug it in the second equation.

$$c^2\frac{4}{9\gamma^2} = \frac{8\pi G}{3}\rho_0\, c^{\frac{4}{3\gamma}-2}$$

$$c^2 = K c^{\frac{4}{3\gamma}-2} \quad , \text{with } K := 6\pi\gamma^2 G\rho_0$$

$$c^{4-\frac{4}{3\gamma}} = K$$

$$c^{\frac{4(3\gamma-1)}{3\gamma}} = K \implies c = K^{\frac{3\gamma}{4(3\gamma-1)}} = \left(6\pi G\gamma^2\rho_0\right)^{\frac{3\gamma}{4(3\gamma-1)}}$$

Therefore the solution to the Friedmann equation is

$$a(t) = \left[\left(6\pi G\gamma^2\rho_0\right)^{\frac{3\gamma}{4(3\gamma-1)}}\right] t^{\frac{2}{3\gamma}}$$

$$\rho(t) = \frac{\rho_0}{a(t)^{3\gamma}} = \left[\rho_0\left(6\pi G\gamma^2\rho_0\right)^{\frac{4(3\gamma-1)}{9\gamma^2}}\right]\frac{1}{t^2}$$

If $p = -\rho c^2$, we have to solve the equations from scratch (why?).

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\cancel{\rho} - \cancel{\frac{\rho c^2}{c^2}}\right) = 0, \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3} \implies \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_0}$$

$$\dot{\rho} = 0 \implies \rho = \text{const.} = \rho_0.$$

$$\frac{da}{a} = \sqrt{\frac{8\pi G}{3}\rho_0}\, dt \implies \ln\left(\frac{a(t)}{a_0}\right) = \sqrt{\frac{8\pi G}{3}\rho_0}\,(t-t_0)$$

$$a(t) = a_0 \exp\left[\left(\frac{8\pi G}{3}\rho_0\right)^{1/2}(t-t_0)\right]$$