RADIATION PHYSICS : overview

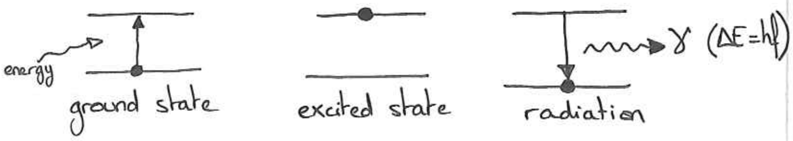
1. Radiation

Heat transfer happens through conduction (between objects in physical contact), convection (through fluid motion), and radiation.

Radiation is a transfer of heat by emission of electromagnetic (EM) energy. The intermediate medium doesn't necessarily participate to the transfer (for exple, the Sun heats the Earth, but the medium in between has a temperature lower than Earth temperature).

- → emission (conversion of source energy into EM energy)
- → transmission (propagation of waves, maybe some absorption)
- → reception (conversion of EM radiation into thermal energy, ie. absorption)

Radiation comes from an electronic transition between 2 energy states of a molecule or an atom



From an energy state $\mathcal{E}$ to an energy state $\mathcal{E} - \Delta\mathcal{E}$ we have emission of a radiation of frequency f and energy hf, where h is Planck's constant ($h = 6{,}62 \cdot 10^{-34}$ J.s).

$[E] = J \quad [h] = J.s \quad [f] = s^{-1}$

A body at temperature T emits waves of ≠ frequencies, and the quantity and repartition of energy depends on the temperature of the body.

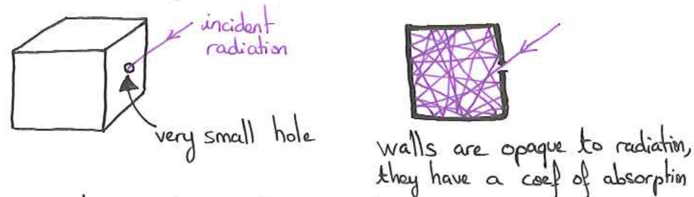
2. Black body

A black body is an object (ideal) absorbing all the EM energy that it receives, with no reflexion, no transmission. Nothing less, nothing more.
It is the perfect thermal radiation emitter, which for a given temperature T, will emit the maximum of energy., **isotropically.**

(→ Can you think of an exemple?
Is a black-hole the perfect black body?
No, because it may not absorb the γ with wavelengths λ higher than its own size!)

→ So it's more of a theoretical concept, but we can approximate many objects as black bodies: stars, black-holes, interstellar medium, and the best black body we know is the CMB (cosmic microwave background).

The perfect black body can be modelised by an absorption cavity:



The incident radiation will be reflected/absorbed upon the walls of the cavity, loosing intensity, until it is completely absorbed.

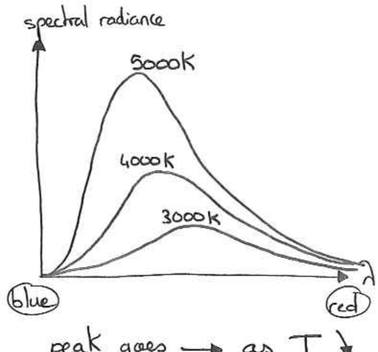
3 important laws:

Black body radiation
- Planck radiation law: radiation energy, temperature, frequency (wavelength)
- Wien displacement law: measure temperature of objects
- Stefan-Boltzmann law: radiation heat transfer

3. Planck's law

The black body spectrum only depends on its temperature.

Planck spectra:



peak goes → as T ↓

(So things emitting in the IR are colder than things emitting in the UV for exple.)

(a human body emits in the IR (cf. IR cameras), fire emits in visible, CMB emits in microwaves (2.7 K) (cold))

Planck's law describes the power radiated per unit area of emitting surface of the body, per unit solid angle that the radiation is measured over, per unit frequency, by a black body at temperature T:

$$\boxed{I(f,T) = \frac{2hf^3}{c^2} \frac{1}{e^{\frac{hf}{k_B T}} - 1}}$$

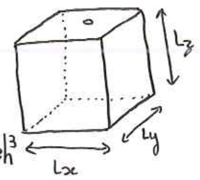
$$\begin{cases} k_B = \text{Boltzmann constant} \\ h = \text{Planck constant} \\ c = \text{speed of light} \end{cases}$$

→ Where does it come from?

Let's consider our cube, the absorbing cavity, model of the black body. We first want to determine the spectral energy density within the cavity.

Heisenberg uncertainty principle:

$\Delta x . \Delta p \geqslant h$



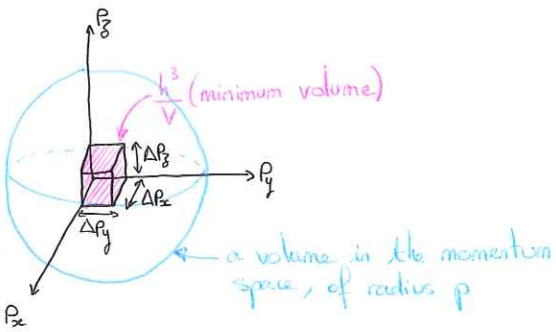
→ in 3D: $\Delta x \Delta y \Delta z . \Delta p_x \Delta p_y \Delta p_z \geqslant h^3$

If the photon is in the box and we don't know its position, then the uncertainty on the position is the length of the box: $\Delta x = L_x$, $\Delta y = L_y$, $\Delta z = L_z$.

$$\Rightarrow \Delta p_x \Delta p_y \Delta p_z \geqslant \frac{h^3}{L_x L_y L_z} = \frac{h^3}{V} = \underline{\underline{\text{minimum}}}.$$

volume of the cavity

So, all the states that are possible in the momentum space can be represented by a cube $\Delta p_x \Delta p_y \Delta p_z$:



In other words, in my box, my photon has a maximum λ that is the size of the box. So since $E = hf = \frac{hc}{\lambda}$, it has a minimum energy, and its energy is its momentum (photons have zero rest mass). The cube is the minimum energy, and each cube can be occupied by 2 photons (2 polarization states), ie. degeneracy factor).

The sphere is a space of a certain energy, now how many cubes of minimum energy can we fit inside?

N = total nb of photons = $\frac{\text{volume sphere}}{\text{volume cube}}$ × degeneracy

$$N = \frac{\frac{4}{3}\pi p^3}{\frac{h^3}{V}} \times 2 \Rightarrow \frac{N}{V} = \frac{8\pi p^3}{3h^3}$$

For a photon, we know that $E = hf$ and $p = \hbar k = \frac{hf}{c}$ (wave vector)

$$\Rightarrow \frac{N}{V} = \frac{8\pi h^3 f^3}{3h^3c^3} \Rightarrow \frac{N}{V} = \frac{8\pi f^3}{3c^3} = \text{density of photons per unit volume}.$$

Now Planck's formula is per unit frequency. So how many photons are contained between f and $f + \Delta f$?

$$N_f = \frac{\partial}{\partial f}\left(\frac{N}{V}\right) = \frac{\partial}{\partial f}\left(\frac{8\pi f^3}{3c^3}\right) = \frac{8\pi f^2}{c^3} = \text{density of photons per unit volume per unit frequency}.$$

Then we can compute the energy density of photons per unit volume per unit frequency:

$$\mathcal{E}(T) = N_f \times \underbrace{\text{energy of photon}}_{= hf} \times \underbrace{\text{probability that the state is occupied by the photon}}_{= \text{Bose-Einstein relation}}$$

$$p(f) = \frac{1}{e^{\frac{hf}{k_BT} - \mu} - 1}$$

μ: chemical potential $= 0$ for photon

$$\Rightarrow \mathcal{E}(T) = \frac{8\pi f^2}{c^3} \times hf \times \frac{1}{e^{\frac{hf}{k_BT}} - 1}$$

$$\mathcal{E}(T) = \frac{8\pi f^3 h}{c^3} \frac{1}{e^{\frac{hf}{k_BT}} - 1}$$

↓

energy density per unit volume: $J.m^{-3}s$

if we multiply it by the velocity at which the energy is moving (c) : $J.m^{-3}.m.s^{-1}s = J.m^{-2}$ [redacted] = power intensity by unit area ☺ (cf. Planck formula).

Also, since the radiation is the same in all directions, we can divide by the total solid angle (4π), to have it per unit solid angle.

$$\Rightarrow I(f,T) = \frac{\mathcal{E}(T) \times c}{4\pi}$$

$$\boxed{I(f,T) = \frac{2hf^3}{c^2} \frac{1}{e^{hf/k_BT} - 1}}$$

We can also write it in terms of λ:

$$f = \frac{c}{\lambda} \Rightarrow df = -\frac{c}{\lambda^2} d\lambda$$

$I_f |df| = I_\lambda |d\lambda|$ (absolute values because small variation indep. of the direction)

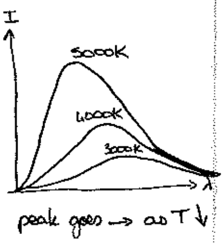
$$\Rightarrow I_\lambda = I_f \frac{|df|}{|d\lambda|} = \frac{2hc^3}{c^2\lambda^3} \frac{c}{\lambda^2} \frac{1}{e^{hc/\lambda k_BT} - 1}$$

$$\boxed{I(\lambda,T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_BT} - 1}}$$

4. Wien's displacement law

The black body radiation curve, for $\neq T$, has a peak λ inversely proportional to T:

$$\boxed{\lambda_{max} = \frac{b}{T}} \quad , \quad b = \text{cste}$$



We derive it from Planck's law:

$$I(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

At the peak wavelength, we have $\frac{\partial I}{\partial \lambda} = 0$

$$\frac{\partial I(\lambda,T)}{\partial \lambda} = 0$$

$$\iff -\frac{10hc^2}{\lambda^6} \frac{1}{e^{hc/\lambda k_B T} - 1} + \frac{2hc^2}{\lambda^5} \frac{hc}{\lambda^2 k_B T} \frac{e^{hc/\lambda k_B T}}{\left(e^{hc/\lambda k_B T} - 1\right)^2} = 0$$

$$\iff \underbrace{\frac{2hc^2}{\lambda^5}\left(\frac{1}{e^{hc/\lambda k_B T} - 1}\right)}_{= I(\lambda,T)} \underbrace{\left[-\frac{5}{\lambda} + \frac{hc}{\lambda^2 k_B T} \frac{e^{hc/\lambda k_B T}}{\left(e^{hc/\lambda k_B T} - 1\right)}\right]}_{= 0} = 0$$

$$\implies \frac{hc}{\lambda k_B T} \frac{e^{hc/\lambda k_B T}}{\left(e^{hc/\lambda k_B T} - 1\right)} - 5 = 0$$

We define $x \equiv \frac{hc}{\lambda k_B T}$, and the equation becomes:

$$\frac{x e^x}{e^x - 1} - 5 = 0$$

Then you just need to solve that. We will not do it here, but the numerical solution is:

$$x = 4,965114231$$

$$x = \frac{hc}{\lambda k_B T} \implies \boxed{\lambda_{max}} = \frac{hc}{x k_B T} = \boxed{\frac{b}{T}}$$

with $b = 2,89776829 . 10^{-3}$ m.K (Wien's displacement constant).

5. Stefan-Boltzmann's law

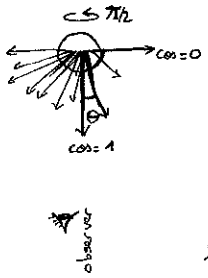
The total power radiated from a black body is:

$$\boxed{\mathcal{E}_{rad} = \alpha T^4} \quad , \quad \alpha = \text{cste}$$

→ We need to integrate Planck's law over all frequencies and over the half-sphere Ω (solid angle):

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$\mathcal{E}_{rad} = \int_0^\infty df \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \; I(f,T) \sin\theta \cos\theta$$



The $\cos\theta$ term comes from the fact that for $\theta = 0$ the energy comes directly towards the observer, who receives it all from that angle. As θ grows, the energy is directed towards a direction with angle θ, and the observer receives less from the energy emitted in that direction, until receiving nothing when $\cos(\theta = \frac{\pi}{2}) = 0$.

$$\mathcal{E}_{rad} = \int_0^\infty I(f,T)\, df \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta \cdot \cos\theta \, d\theta}$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} \frac{\sin(2\theta)}{2} d\theta$$

$$= \int_0^{2\pi} d\varphi \left[-\frac{\cos(2\theta)}{4}\right]_0^{\pi/2}$$

$$= \int_0^{2\pi} \frac{1}{2} d\varphi$$

$$= \pi$$

$$\implies \mathcal{E}_{rad} = \pi \int_0^\infty \frac{2hf^3}{c^2} \frac{1}{e^{hf/k_B T} - 1} df$$

$$= \pi \int_0^\infty \frac{2 k_B^3 T^3}{c^2 h^2} \frac{\frac{h^3 f^3}{k_B^3 T^3}}{e^{\frac{hf}{k_B T}} - 1} df$$

We define $x \equiv \frac{hf}{k_B T}$ → $dx = \frac{h}{k_B T} df$

$$\implies \mathcal{E}_{rad} = \frac{2\pi k_B^4 T^4}{c^2 h^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{= \frac{\pi^4}{15}}$$

(challenging integral, see result in the book p.17)

$$\implies \mathcal{E}_{rad} = \frac{2\pi^5 k_B^4 T^4}{15 c^2 h^3}$$

$$\hbar = \frac{h}{2\pi}$$

$$\implies \boxed{\mathcal{E}_{rad}} = \frac{k_B^4 \pi^2}{60 c^2 \hbar^3} T^4 = \boxed{\alpha T^4} , \boxed{\alpha = \frac{k_B^4 \pi^2}{60 c^2 \hbar^3}}$$