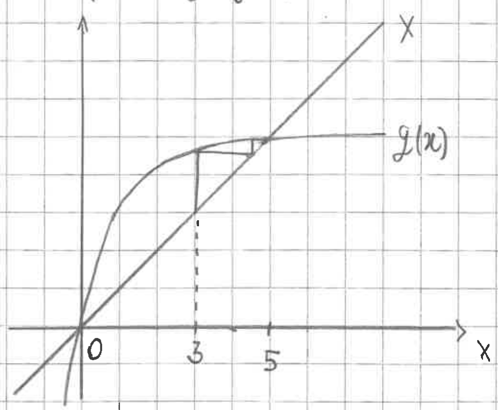
We can solve the equation $\frac{xe^x}{e^x-1}-5=0$ numerically, only with a calculator (you can do it in Python, if you like).

$x = 5(1-e^{-x}) = g(x)$



$g(3) \simeq 4.751064658$

$g[g(3)] \simeq 4.956787555$

$g[g[g(3)]] \simeq 4.964822536$

At the eighth step we get

$x = 4.965114232$

This is called the direct iteration method.

$\frac{xe^x}{e^x-1} - 5 = 0 \ , x \in \mathbb{C}$ To solve this equation, it is necessary to use the Lambert W function, also called omega function or product logarithm.

$xe^x - 5(e^x - 1) = 0$

$xe^x = 5(e^x-1) =: f(x) \implies x = f^{-1}(xe^x) =: W(xe^x)$

We can now define $z := xe^x$ and write

$$ze^{-x} = W(z) \ , \text{ but } \ x = W(z) \ , \text{ therefore}$$

$$z = W(z)e^x = W(z)e^{W(z)}$$

We then define the Lambert W function by the relation

$$z = W(z)e^{W(z)}.$$

Unfortunately, it cannot be written in terms of elementary functions...

For real numbers, it is defined for $x \geqslant -\frac{1}{e}$ and it is double-valued on $\left(-\frac{1}{e}, 0\right)$.

Since ze^z is not injective, W is multi-valued except at $z=0$.

The solution ~~of~~ our equation is

$$x = 5 + W\left(-5e^5\right) \simeq 4.9651142317..$$

The derivative of the Lambert function is

$$\frac{dW}{dz} = \frac{1}{z + e^{W(z)}} \ , \text{ for } z \neq -\frac{1}{e}.$$

W is not differentiable at $z = -\frac{1}{e}$.

This is shown by taking the derivative of the defining equation.

$$\frac{d}{dz}z = \frac{d}{dz}\left(W(z)\, e^{W(z)}\right)$$

$$1 = \frac{dW}{dz}e^{W(z)} + W\frac{dW}{dz}e^{W(z)} = \frac{dW}{dz}e^{W(z)}\left(1 + W(z)\right)$$

$$\left[1 + W(z)\right]\frac{dW}{dz} = e^{-W(z)} \ , \text{ but } \ e^{-W(z)} = \frac{W(z)}{z} \ , \text{ so}$$

$$z\left[1 + W(z)\right]\frac{dW}{dz} = W(z)$$

We have the two formulas,

$$\frac{dW}{dz} = \frac{e^{-W(z)}}{1 + W(z)} = \frac{1}{e^W + We^W} = \frac{1}{z + e^W} \ , \quad \frac{dW}{dz} = \frac{W}{z(1+W)}$$

The first is valid for $z \neq -\frac{1}{e}$, the second for $z \notin \left\{0, -\frac{1}{e}\right\}$.

$$\int \frac{x^3}{e^x - 1} dx = -\frac{x^4}{4} + x^3 \log(1-e^x) + 3x^2 Li_2(e^x) - 6x Li_3(e^x) + 6 Li_4(e^x)$$

$Li_n(z)$ is the polylogarithm or Jonquière's function of order n, $z \in \mathbb{C}$

In quantum statistics, it appears as the closed form of some integrals of the Fermi-Dirac distribution or the Bose-Einstein one.

Only for special values of n it reduces to elementary functions.

The definition is

$$Li_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad s \in \mathbb{C}, \ |z| < 1,$$

and it can be extended to $|z| \geq 1$ by analytic continuation.

$$Li_1(z) = -\ln(1-z).$$

An equivalent definition is

$$Li_{s+1}(z) := \int_0^z \frac{Li_s(t)}{t} dt, \quad \text{with} \quad Li_0(z) = \frac{z}{1-z}.$$

Note that $s \in \mathbb{C}$, so we also have

$$Li_{-1}(z) = \frac{z}{(1-z)^2}.$$

In terms of the FD and BE distributions, we get,

BE: $$Li_s(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\frac{e^t}{z} - 1} dt,$$ which converges for $Re(s) > 0$ and all z except for $z \in \mathbb{R}$ and $z \geq 1$.

FD: $$-Li_s(-z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t/z + 1} dt,$$ which converges for $Re(s) > 0$ and all z except for $z \in \mathbb{R}$ and $z \leq -1$.