Lecture 14

Remember $H^2=\left(\frac{\dot a}{a}\right)^2=\frac{8\pi G}{3}\rho(t)$; $\rho(t)=\frac{\rho_0}{a(t)^3}$ $(k=0)$
for matter

Have to solve $\dot a^2 a = \text{const}$

Ansatz: $a(t)=t^q$ $\quad \dot a = q\,t^{q-1} \sim t^{q-1}$

$$\dot a^2 a \sim t^{2(q-1)}\cdot t^q = \text{const} = t^0$$

$$\Rightarrow 2q-2+q=0 \Rightarrow q=\frac{2}{3} \Rightarrow a(t)\sim t^{\frac{2}{3}}$$

For radiation, $\rho_r \sim \frac{1}{a^4}$ similarly $a(t)\sim t^{\frac{1}{2}}=\sqrt{t}$

Age of the Universe

Matter domination $(\rho_m \gg \rho_r)$ $a\sim t^{2/3}$

$$H(t)=\frac{\dot a}{a}=\frac{\frac{2}{3}t^{-1/3}}{t^{2/3}}=\frac{2}{3t}\ ;\ t=t_0 \Rightarrow H=H_0=70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}\sim 2.3\cdot 10^{-18}\,\text{s}^{-1}$$

$$H_0=\frac{2}{3t_0} \Rightarrow t_0=\frac{2}{3H_0}=\frac{2}{3\cdot 2.3\cdot 10^{-18}}\,\text{s}=3\cdot 10^{17}\,\text{s}\approx 10^{10}\ \text{years}$$

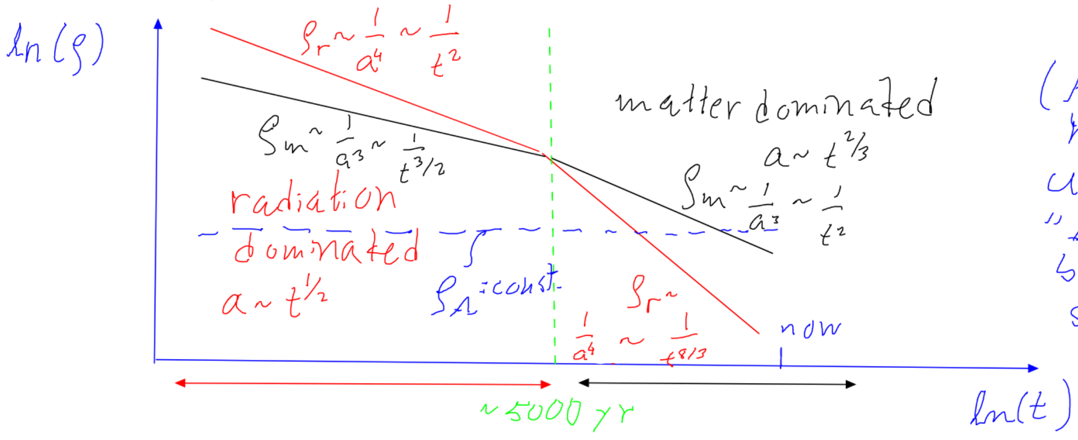
How important is radiation?

At present time, it is very small, CMB ⇒ $\rho_r/\rho_m \sim 10^{-5}$

However, $\frac{\rho_r}{\rho_m}\sim\frac{\left(\frac{1}{a^4}\right)}{\left(\frac{1}{a^3}\right)}\sim\frac{1}{a}$ ⇒ In earliest univ. $a\to 0$ radiation dominated

Curvature $-\frac{k}{a^2}$ ⇒ completely negligible in early univ.

(Also, present data indicate $k\sim 0$)



(Actually, as we have said, the universe is now "Λ dominated", but this is for very small $z \lesssim 2$)

$$\begin{cases} H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} \quad (\mathrm{I}) \text{ Friedmann} \\ \dot\rho + 3\left(\frac{\dot a}{a}\right)(\rho+p) = 0 \quad (\mathrm{II}) \text{ Fluid} \end{cases}$$

Time derivative of (I):

$$\underbrace{\frac{d}{dt}\left(\frac{\dot a}{a}\right)^2}_{} = \frac{8\pi G}{3}\underbrace{\dot\rho}_{-3\left(\frac{\dot a}{a}\right)(\rho+p)} - k\underbrace{\frac{d}{dt}\left(\frac{1}{a^2}\right)}_{\updownarrow}$$

$$\frac{d}{dt}\left(\dot a^2\cdot\frac{1}{a^2}\right) = \frac{2\dot a\ddot a}{a^2} + \dot a^2\left(-\frac{2\dot a}{a^3}\right) = 2\frac{\dot a}{a}\left[\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2\right]$$

Thus, $2\frac{\dot a}{a}\left[\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2\right] = -8\pi G_N\left(\frac{\dot a}{a}\right)(\rho+p) + 2\left(\frac{\dot a}{a}\right)\frac{k}{a^2}$

Divide by $2\left(\frac{\dot a}{a}\right)$:

$$\left[\frac{\ddot a}{a}\underbrace{-\left(\frac{\dot a}{a}\right)^2}_{(\mathrm{I}) \Rightarrow\ = -\frac{8\pi G}{3}\rho + \cancel{\frac{k}{a^2}}}\right] = -4\pi G_N(\rho+p) + \cancel{\frac{k}{a^2}}$$

We finally get

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi G_N}{3}(\rho+3p)}$$

Acceleration equation

The cosmological constant

Einstein wanted (like everybody at that time) a static universe. He noted that in GR it is possible to add a cosmological constant,

Λ (Lambda)

$\Rightarrow$ This modifies the Friedmann equation to

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$\rho_m + \rho_r$

write $\rho_\Lambda = \frac{\Lambda}{8\pi G_N} \Rightarrow \rho = \sum_{i=m,r,\Lambda} \rho_i$

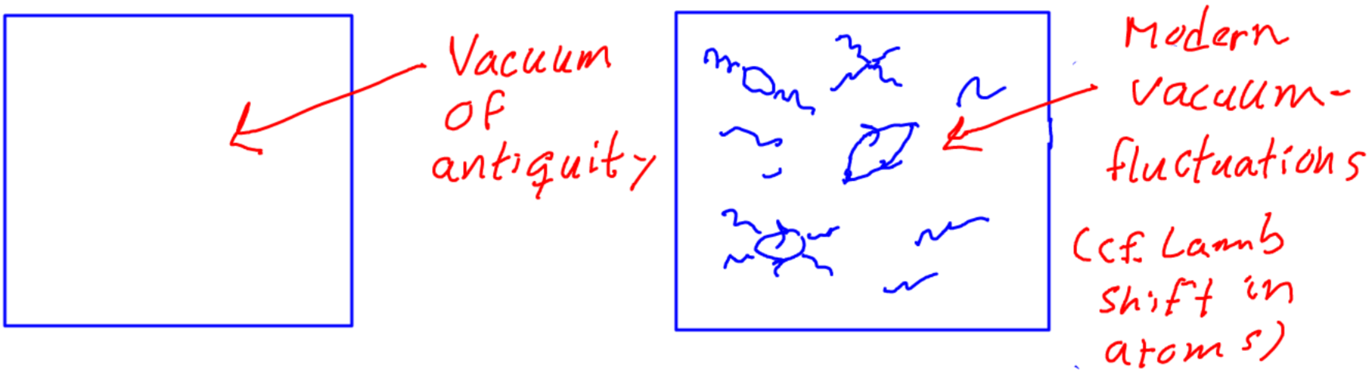
$$\Rightarrow \dots \Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\sum_{i=m,r,\Lambda}(\rho_i + 3p_i) = \frac{-4\pi G_N}{3}\sum_i \rho_i(1+3w_i)$$

$= w_i \cdot \rho_i$

(see formula sheet)

$\Rightarrow$ deceleration for mass and radiation; however $\rho_\Lambda + 3p_\Lambda < 0$!

Modern interpretation: Λ is energy of the vacuum



Vacuum of antiquity

Modern vacuum-fluctuations

(cf. Lamb shift in atoms)

We had $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$ (constant)

and $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{k}{a^2}$

Equation of state for Λ?

Use fluid eqn., $\rho_\Lambda = \text{const} \Rightarrow \dot{\rho}_\Lambda = 0$

But $\underbrace{\dot{\rho}_\Lambda}_{=0} + 3\left(\frac{\dot{a}}{a}\right)(\rho_\Lambda + p_\Lambda) = 0$

$\Rightarrow \rho_\Lambda + p_\Lambda = 0$; $p_\Lambda = -\rho_\Lambda$

cf. $p = w \cdot \rho \Rightarrow w = -1$ for Λ

If $\rho_\Lambda > 0$ then $p_\Lambda < 0$ Negative pressure!

Λ (or something similar, with $p < 0$) is often called <u>dark energy</u>.

<u>Critical density</u>: suppose $k = 0$

$\Rightarrow H^2(t) = \frac{8\pi G}{3}\rho_c(t) \Rightarrow \boxed{\rho_c = \frac{3H^2(t)}{8\pi G}}$

$t = t_0$ (now) $\Rightarrow \rho_c(t_0) = \frac{3H_0^2}{8\pi G} = \dots = 1.88 \cdot 10^{-26} h^2 \frac{kg}{m^3}$

$(H_0 = h \cdot 100 \text{ km/s/Mpc})$

1 solar mass $m_\odot = 1.989 \cdot 10^{30}$ kg

$1 \text{ Mpc}^3 = (3.09 \cdot 10^{22} m)^3 = 2.9 \cdot 10^{67} m^3$

$$\Rightarrow \rho_c = \underbrace{2.7 h^{-1} \cdot 10^{11} m_\odot}_{\text{typical galaxy mass}} / (\underbrace{h^{-1} \text{Mpc}}_{\text{typical distance between galaxies}})^3$$

Define density parameter $\Omega(t) = \frac{\rho(t)}{\rho_c(t)} =$

$$= \frac{\rho_m(t) + \rho_r(t)}{\rho_c(t) \leftarrow \frac{3H^2(t)}{8\pi G}} = \Omega_m(t) + \Omega_r(t);$$

$$\Omega_\lambda = \frac{\Lambda}{3H(t)^2}, \quad \Omega_k = -\frac{k}{H(t)^2 a^2(t)}$$

$$\Rightarrow H^2(t) = \frac{8\pi G}{3} \underbrace{\rho_c(t) \left[\Omega_m(t) + \Omega_r(t)\right]}_{\rho(t)} + \underbrace{\Omega_k(t) H^2(t)}_{-\frac{k}{a^2}} + \underbrace{\Omega_\Lambda(t) H^2(t)}_{\frac{\Lambda}{3}}$$

$$= H^2(t) \left[\overset{\Omega_m + \Omega_r}{\Omega(t)} + \Omega_k(t) + \Omega_\lambda(t) \right]$$

$$\Rightarrow \Omega_m(t) + \Omega_r(t) + \Omega_k(t) + \Omega_\lambda(t) = 1$$

$\Rightarrow$ Not all four can vary independently

$$\Omega_k = -\frac{k}{H^2(t) a^2(t)} \Rightarrow \frac{k}{H^2 a^2} = \Omega_m + \Omega_r + \Omega_\lambda - 1$$

$$\frac{k}{H^2a^2} = \underbrace{\Omega}_{\Omega_m+\Omega_r} + \Omega_\lambda - 1$$

$\Rightarrow$ open universe $(k<0)$ if $\Omega + \Omega_\lambda < \underline{1}$

flat $(k=0)$ $\Omega+\Omega_\lambda = \underline{1}$

closed $(k>0)$ $\Omega+\Omega_\lambda > 1$

Def $\Omega_M = \Omega_M(t_0)$, $\Omega_R = \Omega_R(t_0)$, $\Omega_\Lambda = \Omega_\lambda(t_0)$

we had $\rho_m \sim \frac{1}{a^3(t)} \Rightarrow \frac{\rho_m(t)}{\rho_m(t_0)} = \frac{a_0^3}{a^3}$; $\rho_m(t) = \rho_m(t_0)\left(\frac{a_0}{a}\right)^3$

In similar way, $\rho_{rad}(t) = \rho_{rad}(t_0)\left(\frac{a_0}{a}\right)^4 \leftarrow$ 4 instead of 3 due to redshift

$\rho_k = \rho_k(t_0)\left(\frac{a_0}{a}\right)^2$, $\rho_\Lambda = \rho_\Lambda(t_0)$

$$\Rightarrow H^2(t) = H_0^2\left[\Omega_M\left(\frac{a_0}{a}\right)^3 + \Omega_R\left(\frac{a_0}{a}\right)^4 + \Omega_K\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right]$$

But $\frac{a_0}{a} = (1+z)$ (redshift)

use z instead of (lookback) time as variable

$$\Rightarrow H^2(z) = H_0^2\left[\underset{\substack{\uparrow \\ \sim 0.3}}{\Omega_M}(1+z)^3 + \underset{\substack{\uparrow \\ \sim 10^{-5}}}{\Omega_R}(1+z)^4 + \underset{\substack{\uparrow \\ \sim 10^{-3}}}{\Omega_K}(1+z)^2 + \underset{\substack{\uparrow \\ \sim 0.7}}{\Omega_\Lambda}\right]$$

Measurements: ~ 0.3 $\sim 10^{-5}$ $\sim 10^{-3}$ ~ 0.7

$\Rightarrow$ Today, matter and cosm. const. dominate!

And $\underbrace{\Omega}_{\Omega_M+\Omega_R} + \Omega_\Lambda = 1$ to within 1 permille

$\Rightarrow$ Universe is geometrically <u>flat</u>

Deceleration (or, rather acceleration) parameter

historical name

$a = a(t)$; for $t \approx t_0$ use Taylor expansion:

$$a(t) = a(t_0) + \dot a(t_0)(t-t_0) + \frac{\ddot a}{2!}(t-t_0)^2 + \dots =$$

$$= a(t_0)\left[1 + \underbrace{\frac{\dot a(t_0)}{a(t_0)}}_{=H_0}(t-t_0) + \frac{\ddot a}{a}\frac{(t_0-t)^2}{2} + \dots\right] =$$

$$\equiv a(t_0)\left[1 + H_0(t-t_0) - q_0 H_0^2 (t-t_0)^2 + \dots\right]$$

This defines the deceleration parameter

$$q_0 = -\frac{\ddot a(t_0)}{a(t_0) H_0^2} \qquad \text{(dimensionless)}$$

Acceleration eqn.

Eqn. of state $p = w\rho$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(1+3w)\rho$$

We have multiple "fluids", matter, radiation, Λ, thus

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\sum_i \rho_i (1+3w_i) = -\frac{4\pi G}{3}\sum_i \rho_i (1+3w_i)$$

$$\Rightarrow q_0 = \frac{-\ddot a(t_0)}{H_0^2 a(t_0)} = \frac{4\pi G}{3H_0^2}\sum_i \rho_i(t_0)(1+3w_i)$$

$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G} \Rightarrow$$

Put $\Omega_K \sim 0$ and also $\Omega_R \sim 0$

$$w_\Lambda = -1 \qquad w_M = 0$$

$$\Rightarrow q_0 = \frac{4\pi G}{3H_0^2}\sum_i (1+3w_i)\overbrace{\rho_i(t_0)}^{\equiv \Omega_i \overset{\frac{3H_0^2}{8\pi G}}{\swarrow}\rho_c(t_0)} =$$

$$= \frac{1}{2}\sum_I \Omega_I(1+3w_I) = \frac{1}{2}\left[\Omega_M - 2\Omega_\Lambda\right] =$$

$$= \frac{1}{2}\Omega_M - \Omega_\Lambda$$

$$\Omega_M + \Omega_\Lambda + \cancel{\Omega_K} + \cancel{\Omega_R} = 1$$

neglect

$$\Rightarrow \Omega_M + \Omega_\Lambda = 1 \qquad \Omega_\Lambda = 1 - \Omega_M$$

$$\Rightarrow q_0 \simeq \frac{3\Omega_M}{2} - 1$$

$\Omega_M \sim 0{,}3 \quad \Omega_{stars} \sim 10^{-2} \quad \Omega_B = \Omega_{stars} + \Omega_{gas,dust}$

baryons

$\sim 0{,}05$ (from CMB and big bang nucleosynthesis, see next lecture)

$\Omega_M - \Omega_B \equiv \Omega_{DM} \leftarrow$ Dark (or rather invisible) matter [non-baryonic matter]

$$\boxed{\Omega_{DM} \sim 0.25}$$