# Lecture 13, 2018-10-11



Convention: $\frac{d}{dt} y(t) \equiv \dot{y}(t) \equiv \dot{y}$ (time dependence implicit)

Universe is expanding!

$\Rightarrow$ Recession velocity $v(t) = \frac{d}{dt} r(t) =$

$$= \frac{d}{dt}\left[a(t) \cdot x\right] = \dot{a} \cdot \underbrace{x}_{= r/a} = \frac{\dot{a}}{a} \cdot r$$

physical distance

$\frac{\dot{a}(t)}{a(t)} \equiv H(t)$ Hubble parameter

(simplified notation: $\frac{\dot{a}}{a} = H$)

today: $t = t_0 \Rightarrow H(t_0) \equiv H_0 = \frac{\dot{a}(t_0)}{a(t_0)}$

$H_0$ is the Hubble <u>constant</u>

cepheids $\Rightarrow H_0 \approx (70 \pm 3)$ km/s per Mpc

Other convention (historical): $H_0 = h \cdot 100$ km/s $\text{Mpc}^{-1}$

$\Rightarrow h = 0.70 \pm 0.03$ (see formula sheet)

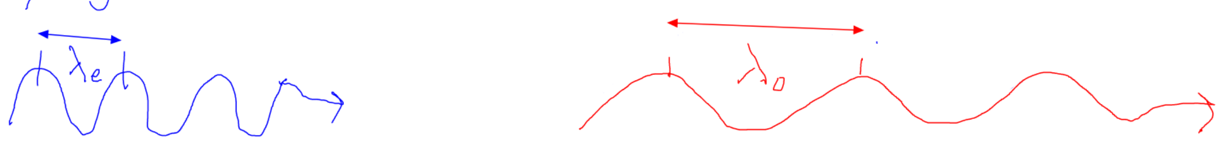
Using general relativity (see Liddle p. 129) one can show

$$\boxed{\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = 1 + z}$$

$\lambda_e$: emitted wavelength

$\lambda_0$: observed —"—

very general formula (valid also for high $z$)

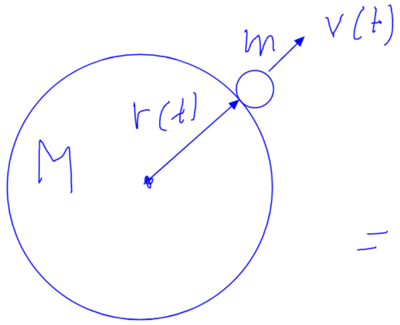



"stretching" of light-waves $\Rightarrow$ red-shift

# The Friedmann equation

(see Liddle 3.1) Newtonian treatment (gives right result!):

Assume homogenous sphere of density $\rho(t)$.
Radius $r(t)$. How will velocity of $m$ (small test mass) change?
$v(t) = \dot{r}(t)$, look at energy situation




Mass within $r$: $M(t) = \frac{4\pi}{3} r^3(t) \cdot \rho(t)$

Pot. energy $V(t) = -\frac{G_N M(t) m}{r(t)} + \underbrace{\text{const}}_{\text{choose } = 0} =$

$= \frac{-4\pi G_N}{3} r^2(t) \rho(t) m$ $\quad G_N$: Newton's constant

Kin. energy $T(t) = \frac{1}{2} m v^2(t) = \frac{1}{2} m \dot{r}^2$; write $r(t) = a(t) \cdot x$

$\dot{r}(t) = \dot{a}(t) \cdot x$

Total energy $U = T + V = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi G_N}{3} a^2 x^2 \rho m =$

$= m(ax)^2 \left[ \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 - \frac{4\pi G_N}{3} \rho \right]$

$\Rightarrow \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho(t) + \frac{2U}{m(ax)^2}$

depend only on time

$\Rightarrow$ This must be indep. of $x$

$\Rightarrow U \sim x^2$; Put $\frac{2U}{mx^2} = -kc^2$ (This is a bit strange, but leads to the GR result)

$$\Rightarrow \boxed{H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho(t) - \frac{kc^2}{a^2}}$$

The Friedmann eqn.

Formula sheet: $\rho \to \rho_m + \rho_r$ (matter + radiation)

Another constant $\frac{\Lambda}{3}$ is added ("dark energy" - cosm. const.)

$\frac{a(t_0)}{a(t)} = \frac{\lambda_0}{\lambda_e} = 1 + z$; $\frac{\rho_m(t)}{\rho_m(t_0)} = \frac{a^3(t_0)}{a^3(t)} = (1+z)^3$ dilution (see later)

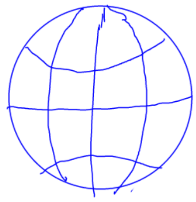
In similar way, $\frac{\rho_r(t)}{\rho_r(t_0)} \sim 1/a^4(t) \sim (1+z)^4$ dilution + redshift

## Friedmann eqn:

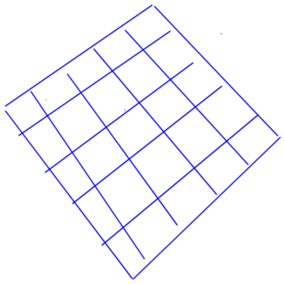
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}(\rho_m + \rho_r) - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$k = \frac{-2U}{m^2c^2x^2}$ determines the geometry of the universe

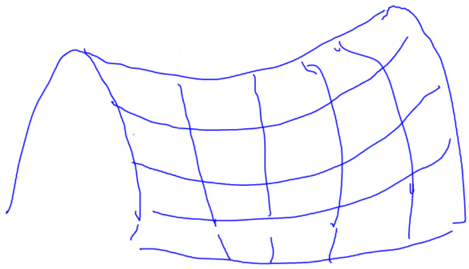
$k>0$ "closed" 2-dim analogy: sphere
finite radius



$k=0$ plane
infinite

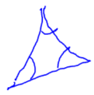


$k<0$
saddle
infinite



sum of angles of triangle on sphere is $>180°$

sum is $=180°$



sum is $<180°$

The distances can be written as

$$d\bar{r}^2 = a^2\left(\frac{dx^2}{1-kx^2} + x^2 d\varphi^2\right)$$

$k=0$ plane
$k=1$ sphere
$k=-1$ saddle

[Check: plane, use polar coordinates $\rho, \varphi$

$$d\bar{r}^2 = d\rho^2 + \rho^2 d\varphi^2; \text{ let } \rho = a\cdot x \Rightarrow d\bar{r}^2 = a^2(dx^2 + x^2 d\varphi^2)$$ ]

Relativity $\Rightarrow$ space-time is defined by

$$ds^2 = c^2dt^2 - a^2\left(\frac{dx^2}{1-kx^2} + x^2 d\varphi^2\right) \quad \text{2D+time}$$

3D + time $\Rightarrow$ $ds^2 = c^2dt^2 - a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$ $\quad d\Omega^2 = \sin^2\theta\, d\theta^2 + d\varphi^2$

From now on, choose $c=1$ (i.e. measure all velocities in units of $3\cdot 10^8$ m/s)
and for the moment put $\Lambda = 0$

$$\Rightarrow \text{Friedmann eqn.} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\underbrace{\rho(t)}_{\rho_m(t)+\rho_r(t)} - \frac{k}{a^2(t)}$$

But, how does $\rho$ depend on time?

Need "fluid equations", can be derived for adiabatic gas:

$$\frac{dE}{dt} + p\frac{dV}{dt} = 0 \quad (\mathrm{I})$$

For unit of co-moving volume $V = \frac{4\pi}{3} a^3 \cdot \underbrace{x^3}_{=1 \text{ (unit volume)}}$ 4

$\frac{dV}{dt} = 4\pi a^2 \dot{a}$  $E = V \cdot \rho$

$\frac{dE}{dt} = 4\pi a^2 \dot{a} \rho + \frac{4\pi}{3} a^3 \dot{\rho}$

$p \cdot \frac{dV}{dt} = p \cdot 4\pi a^2 \dot{a}$; $(I) \Rightarrow \frac{dE}{dt} + p \frac{dV}{dt} =$

$= \cancel{4\pi a^2} \dot{a} \rho + \frac{\cancel{4\pi}}{3} \cancel{a^3} \dot{\rho} + p \cdot \cancel{4\pi a^2} \dot{a} = 0$

Multiply by $\frac{3}{a} \Rightarrow \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0$

n: number density, $\rho = m \cdot n$

this is really $\frac{\rho}{c^2}$

Ideal gas, $p = \frac{1}{3} n \langle \vec{v} \cdot \underbrace{\vec{p}}_{mv} \rangle \sim \rho v^2$. For matter in the universe (in galaxies, clusters, etc) $v/c \sim 10^{-3} \Rightarrow \frac{p}{c^2} \approx 0$

Thus, for matter, $\dot{\rho} + 3 \frac{\dot{a}}{a} \rho = 0$  $(p = 0)$

$\Rightarrow \frac{\dot{\rho}}{\rho} = -3 \frac{\dot{a}}{a} \Rightarrow \ln \rho = -3 \ln a + \text{const}$

Exponentiate! $\Rightarrow \rho = \frac{\rho_0}{a^3}$ ∵ If $M = M_0 = \text{const}$, $\rho = \frac{M_0}{V} \sim \frac{M_0}{a^3(t)}$
(dilution)

If $p \neq 0$; say $p = w \cdot \rho$ (Equation of state)
then in a similar way (do this!) $\frac{\dot{\rho}}{\rho} = -3(1+w)\left(\frac{\dot{a}}{a}\right)$

$\Rightarrow \rho \sim \frac{1}{a^{3(1+w)}}$  Ex: radiation, $p = \frac{\rho c^2}{3} \overset{c=1}{=} \frac{\rho}{3} \Rightarrow w = \frac{1}{3}$

$\Rightarrow \rho_r \sim \frac{1}{a^{3(1+\frac{1}{3})}} = \frac{1}{a^4}$  (redshift, $E = h\nu$; $c = \lambda \cdot \nu \Rightarrow \nu$ decreases and therefore the energy $h\nu$)

As we had guessed!

Look first at $k = 0$ "flat" universe (in 3D)

$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \rho(t)$, where $\rho(t) = \frac{\rho(t_0)}{a^3(t)}$  ($a(t_0) = 1$ chosen)

Thus, $\dot{a}^2 a = \frac{8\pi G_N}{3}\rho_0 = \text{const}$

How to find time dependence of $a$, $a = a(t)$?

Have to solve $\dot{a}^2 a = \text{const}$

Ansatz: $a(t) = t^q \qquad \dot{a} = q t^{q-1} \sim t^{q-1}$

$\dot{a}^2 a = t^{2(q-1)} \cdot t^q = \text{const} = t^0$

$\Rightarrow 2q - 2 + q = 0 \Rightarrow q = \frac{2}{3} \Rightarrow a(t) \sim t^{\frac{2}{3}}$

For radiation, $\rho_r \sim \frac{1}{a^4}$ similarly $a(t) \sim t^{\frac{1}{2}} = \sqrt{t}$

## Age of the Universe

Matter domination $(\rho_m \gg \rho_r)$ $a \sim t^{2/3}$

$H(t) = \frac{\dot{a}}{a} = \frac{\frac{2}{3}t^{-1/3}}{t^{2/3}} = \frac{2}{3t}$ ; $t = t_0 \Rightarrow H = H_0 = 70\,\text{km}\,s^{-1}\,\text{Mpc}^{-1} \sim 2.3 \cdot 10^{-18}\,s^{-1}$

$H_0 = \frac{2}{3t_0} \Rightarrow t_0 = \frac{2}{3H_0} = \frac{2}{3 \cdot 2.3 \cdot 10^{-18}}\,s = 3 \cdot 10^{17}\,s \approx 10^{10}$ years

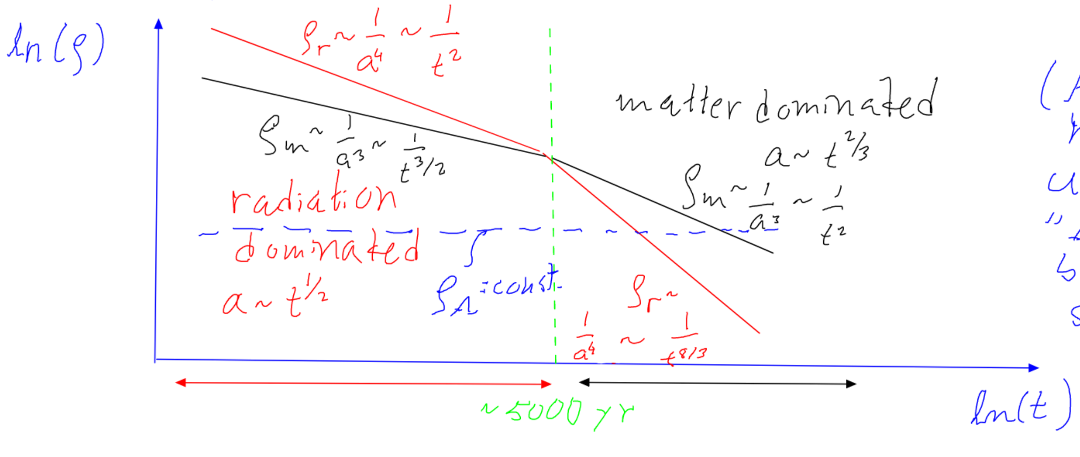
How important is radiation?

At present time, it is very small, CMB $\Rightarrow \rho_r / \rho_m \sim 10^{-5}$

However, $\frac{\rho_r}{\rho_m} \sim \frac{(\frac{1}{a^4})}{(\frac{1}{a^3})} \sim \frac{1}{a} \Rightarrow$ In earliest univ. $a \to 0$ radiation dominated

Curvature $-\frac{k}{a^2} \Rightarrow$ completely negligible in early univ.

(Also, present data indicate $k \sim 0$)




(Actually, as we have said, the universe is now "$\Lambda$ dominated", but this is for very small $z \lesssim 2$)