Lecture 4

Bound states

- Generic properties of bound states
- Bound states
  - electromagnetic
  - strong
- Strong force potential

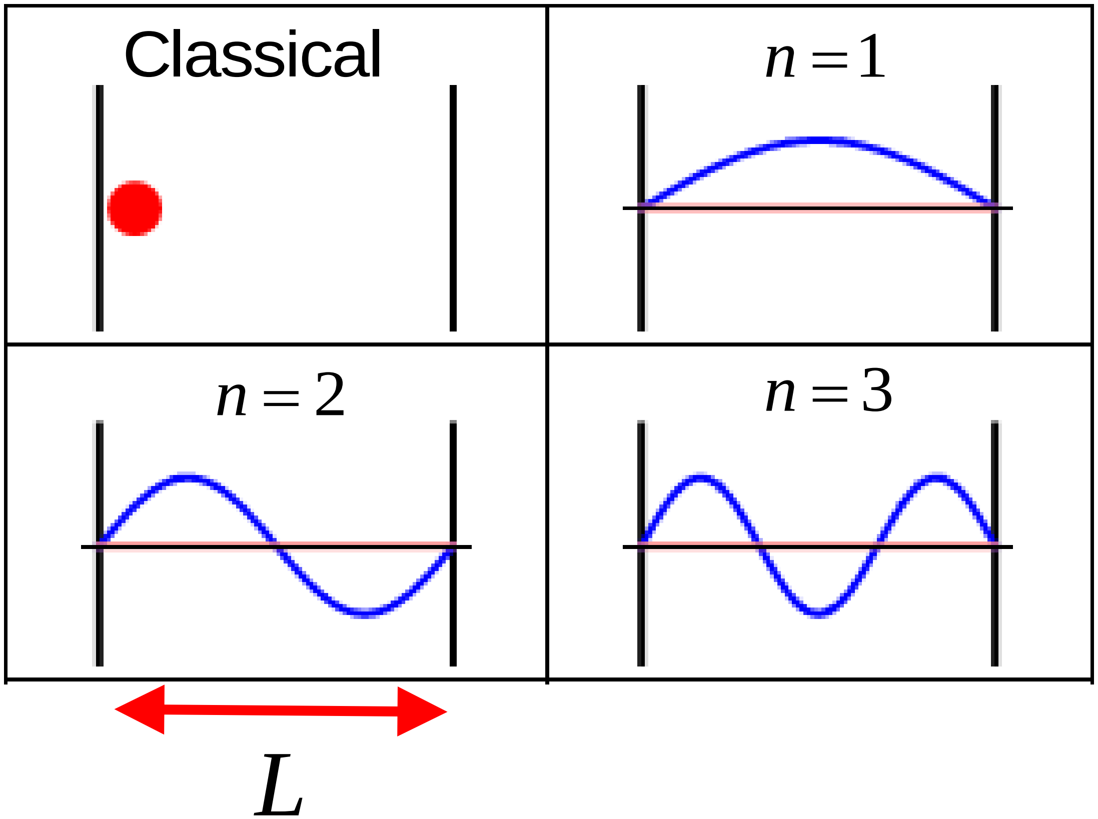
Bound states

- A particle in an infinite potential well
- An electron and a proton in a hydrogen atom
- Neutrons and protons in a nucleus
- Quarks in a proton

Three generic properties of bound states

Particle in an infinite potential well

$$\psi_n = A\sin p_n x = A\left(\frac{e^{i\frac{p_n x}{\hbar}} - e^{-i\frac{p_n x}{\hbar}}}{2i}\right)$$



Standing waves built from

travelling waves $e^{i\frac{p_n x}{\hbar}}, e^{-i\frac{p_n x}{\hbar}}$

Quantised energy : $E_n = \frac{p_n^2}{2m}$

Property #1

Energies of bound states are quantised !

Question

- What happens if the well is reduced in width ?
- Does the particle have more or less energy ?

$$\Delta x \Delta p \sim \hbar \quad ; \ \Delta x \sim L \quad \Rightarrow \quad \Delta p \sim \frac{\hbar}{L}$$

Shrink $L \Rightarrow$ Increase Δp

Eg ground state: $\psi_1 = A \sin p_1 x = A \left(\frac{e^{i\frac{p_1 x}{\hbar}} - e^{-i\frac{p_1 x}{\hbar}}}{2i} \right)$

Two travelling waves, momenta $= -\vec{p}_1, \vec{p}_1$

$$\Rightarrow \Delta p = p_1 - (-p_1) \sim p_1$$

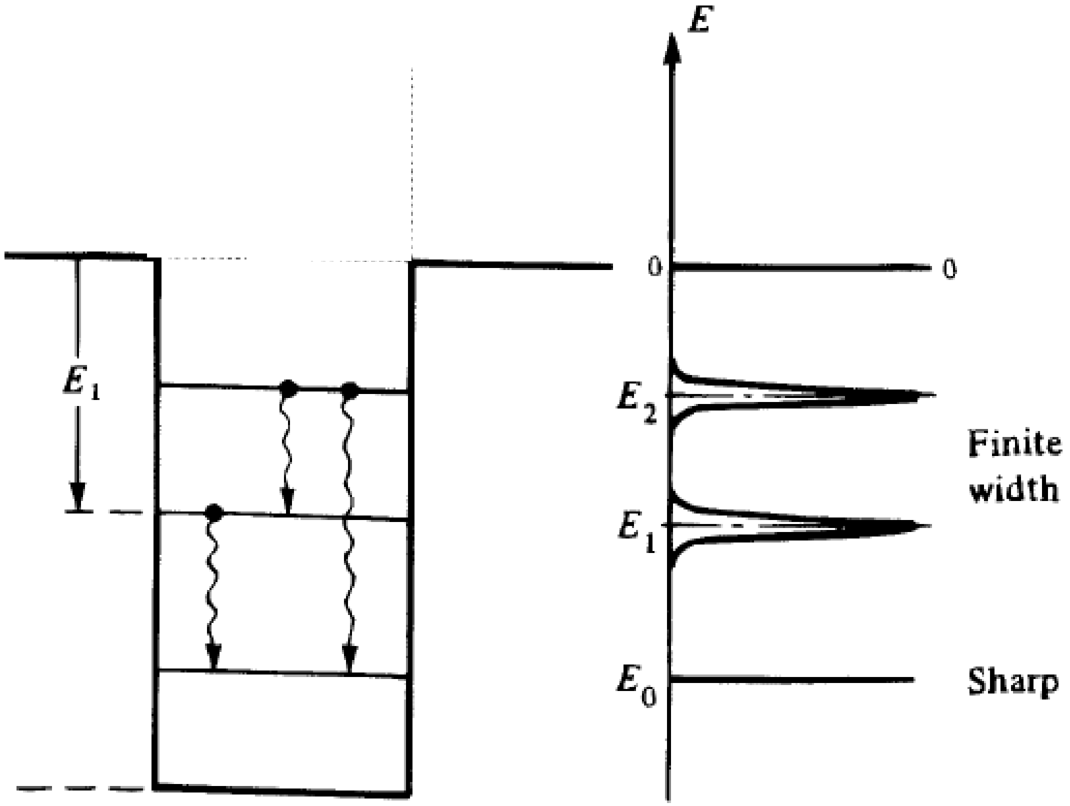
$$\Rightarrow p_1 \sim \frac{\hbar}{L}$$

$\Rightarrow$ Property #2

Particle momentum in a bound state increases as size decreases !

Excited states

Nature wants a bound state to fall to the ground state.



$\Delta E \Delta t \sim \hbar$

Lifetime of state $\tau \sim \Delta t$

Width of state: $\Delta E \sim \frac{\hbar}{\Delta t}$

Property #3

Excited states are unstable :

(1) decay by the emission of a particle, eg photon.

(2) have finite liftetimes and a finite width $\Delta E \sim \frac{\hbar}{\Delta t}$

The ground state can also be unstable.

Generic properties of bound states

- Quantised energy levels
- Increased momentum of constituents as size is reduced
- Unstable excited states with an energy width.

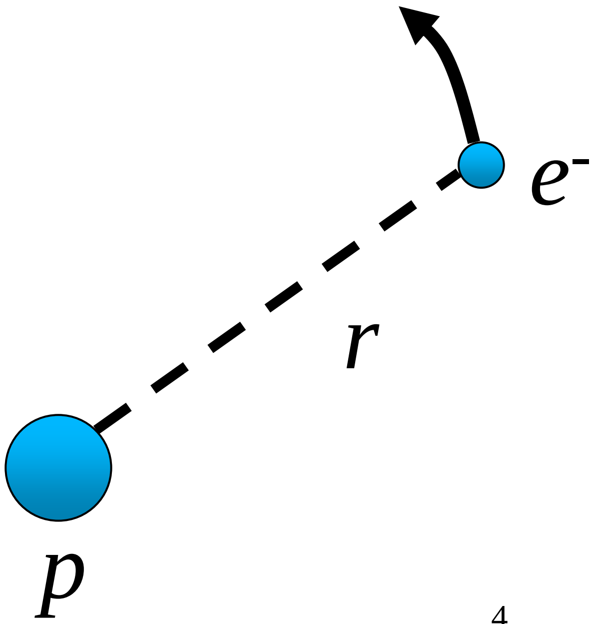
- Details, eg energy values, momenta, widths depend on the system and the force.

Hydrogen atom

Consider an electron around a proton (at the origin).

$$-\frac{1}{2m_e}\nabla^2\Psi(r)+V(r)\Psi = E\Psi$$

$$V(r)=\frac{-e^2}{4\pi\varepsilon_0 r} \quad ; \; m_e = \text{electron mass}$$



Solution wave function of electron:

$$\Psi_{nlm} = R_{n,l}(r)Y_l^m\left(\theta,\phi\right)$$

Bound state $\Rightarrow$ electron energy quantised: $E_n = \dfrac{m_e e^4}{32\left(\pi\varepsilon_0\right)^2 n^2}$

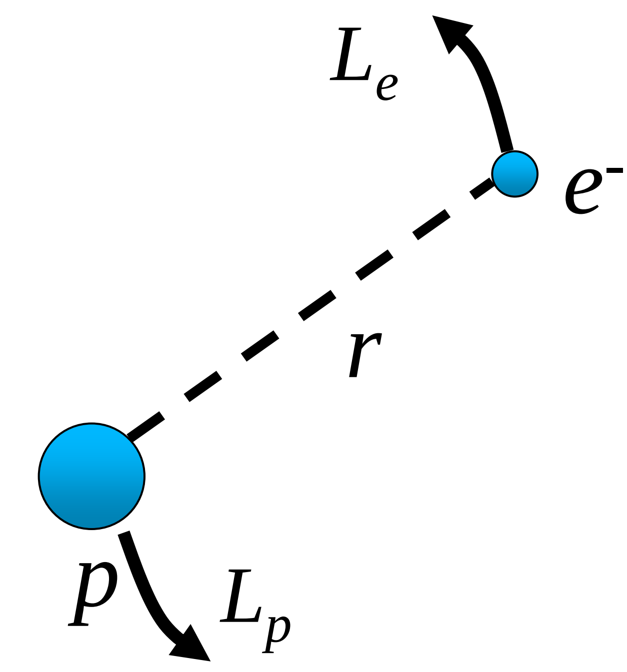
Electron orbital angular momentum quantised:

$$L^2 = l\left(l+1\right) \quad L_z = m_l \quad (-l \le m_l \le l)$$

Its also quite wrong for (at least) two reasons.

The first reason – two body motion

The proton also has a (tiny) angular momentum.
We should talk about both particles motion about their common centre of mass and orbital angular momentum of the *system*.

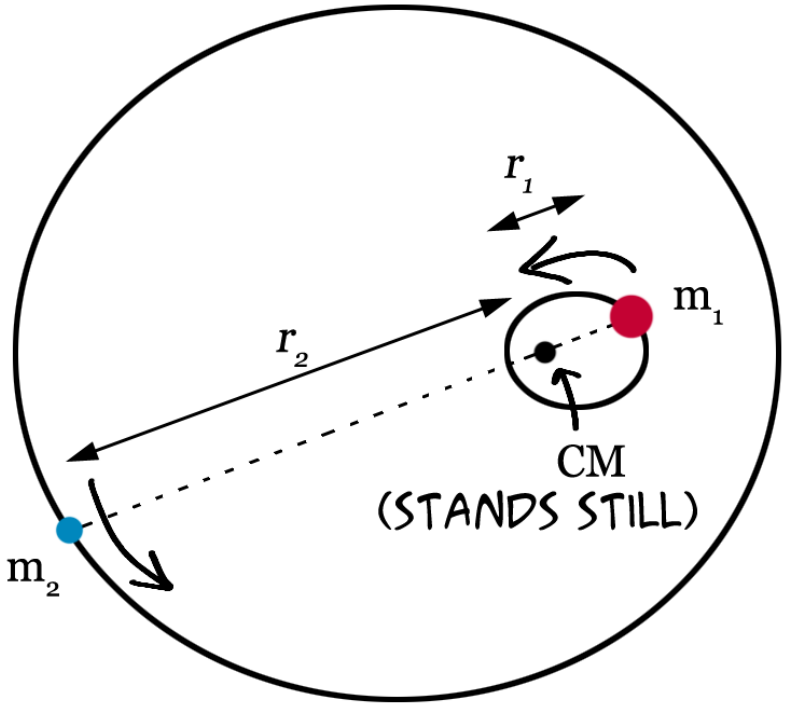


$$-\frac{1}{2m_e}\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial z_2^2}\right]$$
$$+V(r)\Psi\left(x_1,y_1,z_1,x_2,y_2,z_2\right)=E\Psi\left(x_1,y_1,z_1,x_2,y_2,z_2\right)$$

A two body system $\rightarrow$ reduce to a one body problem

$\left[m_1, m_2\right.$ rotating around a common centre-of-mass$\left.\right]$ to a one body problem.

Orbit of a reduced mass $\mu=\frac{m_1 m_2}{m_1+m_2}$ around origin at distance r.



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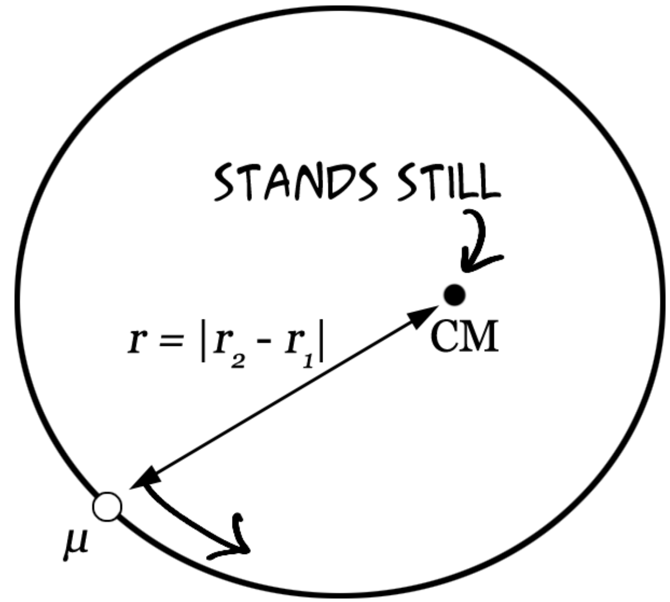
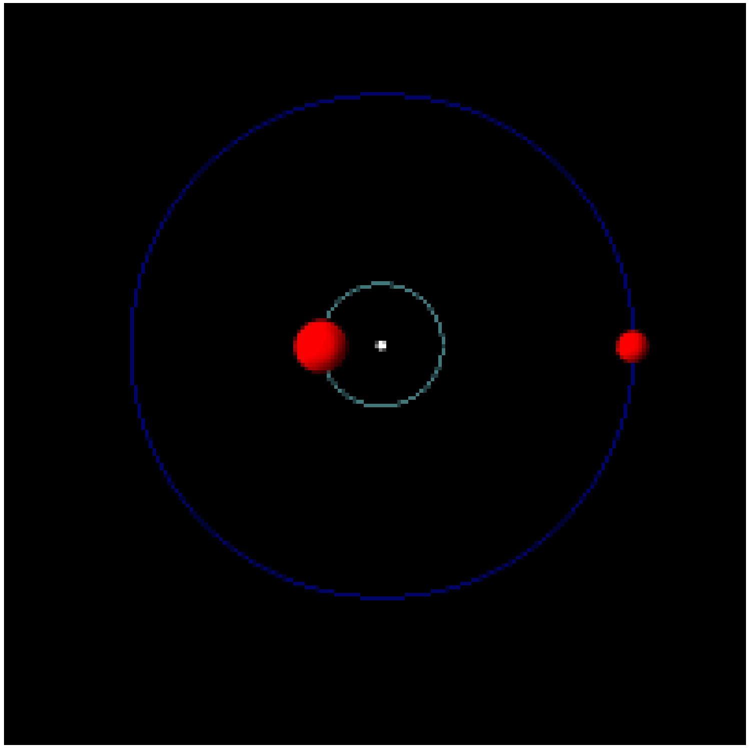


INTERACTING
TWO-BODY SYSTEM

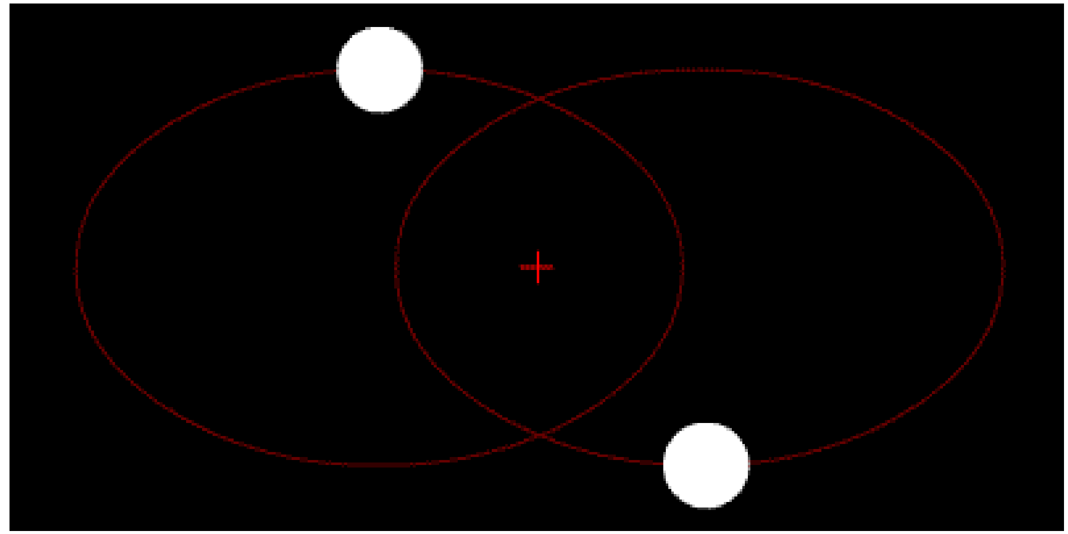
CENTER OF MASS-
REDUCED MASS SYSTEM

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Motion around the centre-of-mass3,2

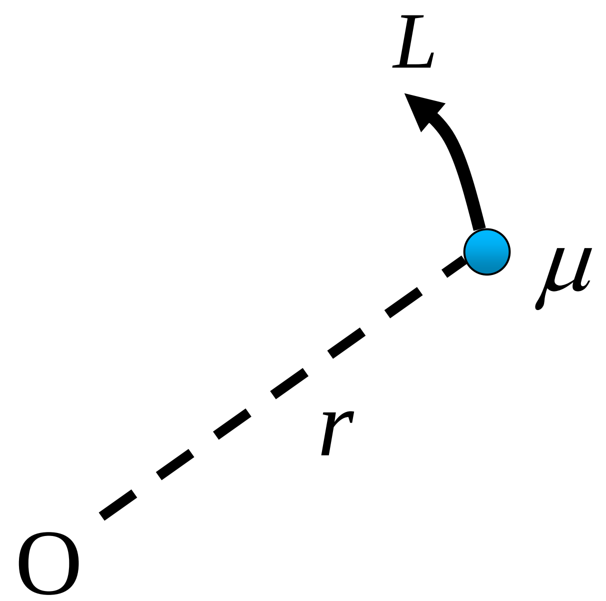


A light and a heavy particle in a bound state.



Two particles of the same mass in a bound state.

Using reduced mass



Use reduced mass $\mu = \dfrac{m_e m_p}{m_e + m_p}$

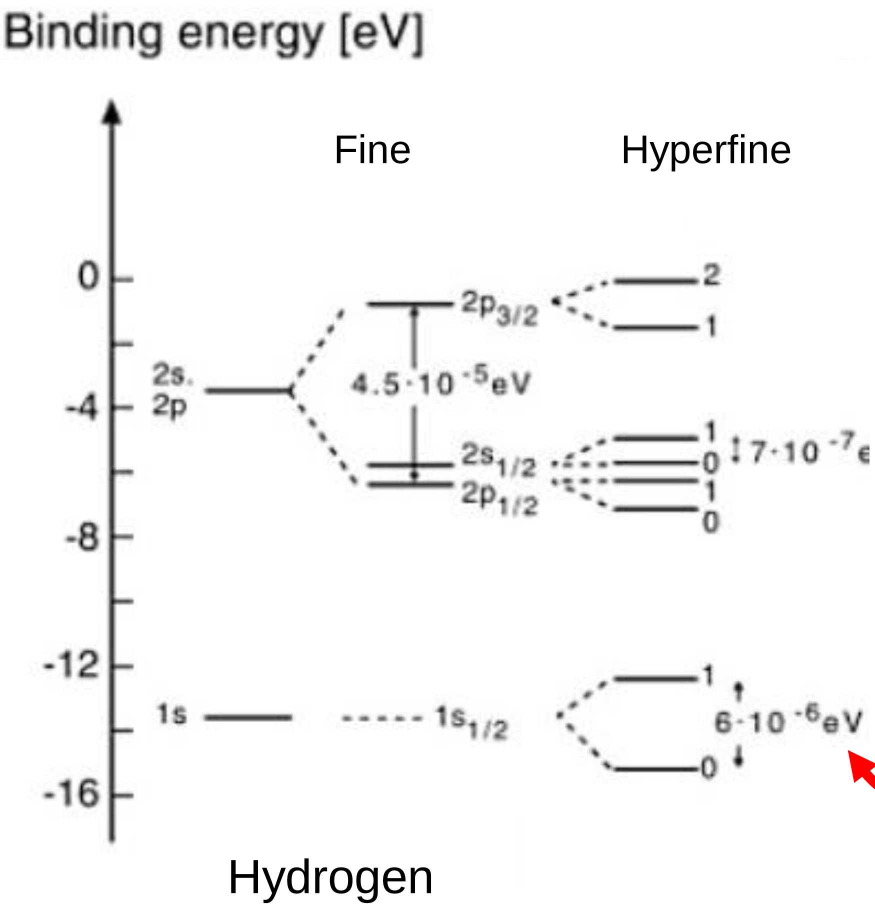
$$-\frac{1}{2\mu}\nabla^2\Psi(r) + V(r)\Psi = E\Psi$$

Quantised energy levels: $E_n = -\dfrac{\mu e^4}{32\left(\pi\varepsilon_0\right)^2 n^2}$

Angular momentum quantised: $L^2 = l\left(l+1\right) \quad L_z = m_l \quad (-l \le m_l \le l)$

Of course very little effect since $\mu \sim \dfrac{m_e m_p}{m_p} \sim m_e$

The second reason – fine and hyperfine structure



Fine structure:

(1) relativistic correction $T = \frac{p^2}{2m} \rightarrow T = \frac{p^2}{2m} - \frac{p^4}{8m^3}$ (9.29)

Perturbative piece to Hamiltonian $H_{rel} = -\frac{p^4}{8m^3}$ (9.30)

(2) spin-orbit coupling: $H_{so} = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{2m_e^{\ 2}}\right)\left(\frac{\vec{L}\bullet\vec{S}}{r^3}\right)$ (9.31)

Frame where electron is stationary and nucleus orbits it.
Proton orbit generates a magnetic field which the electron's magnetic momentum interacts with.

(3) Lamb shift (QED)

Hyperfine structure:

Magnetic moments of proton and electron interact (spin-spin)

$$\Delta E = \frac{\beta}{m_e m_p}(\vec{S}_e \bullet \vec{S}_p)$$

Most relevant for this lecture! Breaks the degeneracy of the ground state. Study simple $l = 0$ states.

Positronium

Look for a model of a $\bar{q}q$ system.

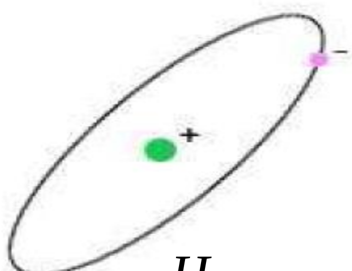
Try positronium e^+e^- - analogous to hydrogen e^-p

In classical language the e^+ and e^- rotate around their centre-of-mass.

Use analogy and earlier mathematics to work out energy levels in positronium.

$$\mu_{pos} = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2} \qquad \mu_H \approx m_e$$

$$H:\ E_n^H = \frac{-\mu_H e^4}{32(\pi\varepsilon_0)^2 n^2} \quad , \quad E_n^{pos} = \frac{-\mu_{pos} e^4}{32(\pi\varepsilon_0)^2 n^2} \approx \frac{-\frac{\mu_H}{2} e^4}{32(\pi\varepsilon_0)^2 n^2} \approx \frac{E_n^H}{2}$$



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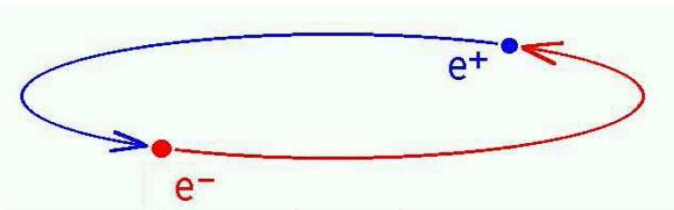
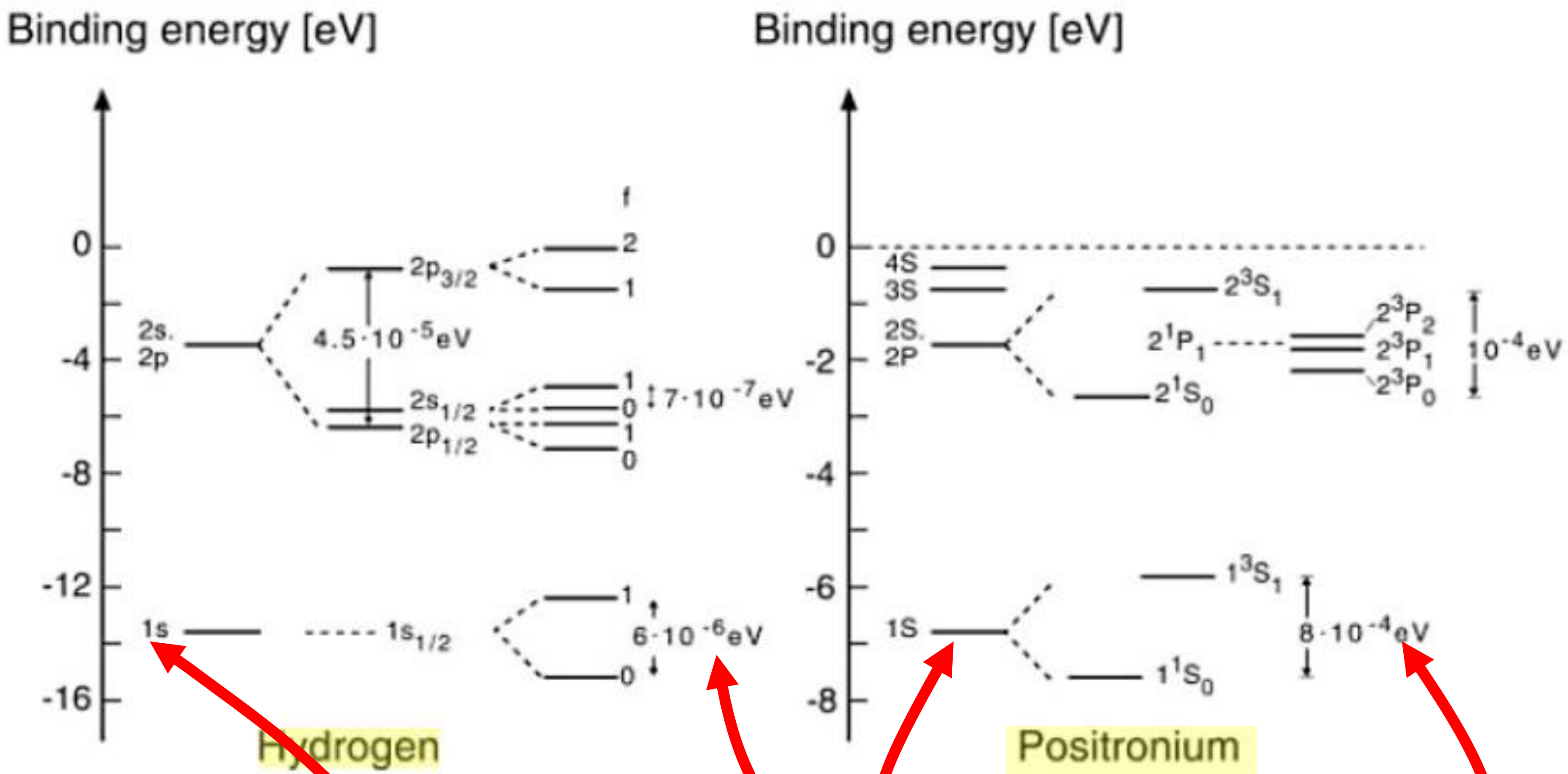


positronium

Positronium energy levels



Energy level hydrogen $\sim\ 2\times$ energy level positronium (reduced mass).

Hyperfine/spin-spin structure:

$$\Delta E_H = \frac{\beta}{m_e m_p} \quad ; \Delta E_P = \frac{\beta'}{m_e m_e}(\vec{S}_e \bullet \vec{S}_p) \Rightarrow \Delta E_H << \Delta E_P$$

Not tiny/hyperfine for positronium since $m_p >> m_e$.

More on positronium

State can have spin and orbital angular momentum.

Lowest two states:

$^3S_1 \ (n=1)$ -orthopositronium ; $^1S_0 \ (n=1)$ - parapositronium

Spectroscopic formalism:

$(^{2s+1}L_j$; $j=$ total ang.momentum quantum number

$s=$ spin quantum number,

$2s+1=$ total number of spin states.

$L \equiv S, P, D..$ (standard notation).

$S:(l=0), \quad P:(l=1), \ D:(l=2)$

State		J
n=1	$^1S_0$	0
	$^3S_1$	1
n=2	$^1S_0$	0
	$^3S_1$	1
	$^1P_1$	1
	$^3P_2$	2
	$^3P_1$	1
	$^3P_0$	0

The strong force

- Apply what we’ve learned for hadrons to understand how the strong force works.

Hadrons and the strong force

The electromagnetic force is easy..

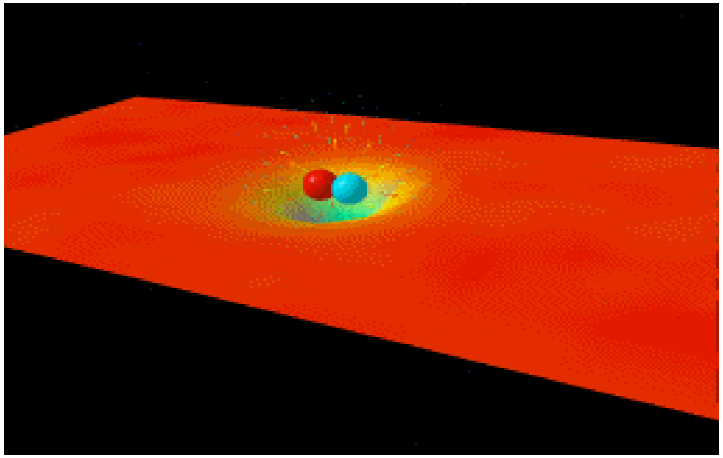
We understand it at macroscopic level and apply it within quantum mechanics.

$\Rightarrow$ Predict atomic spectra, positronium....

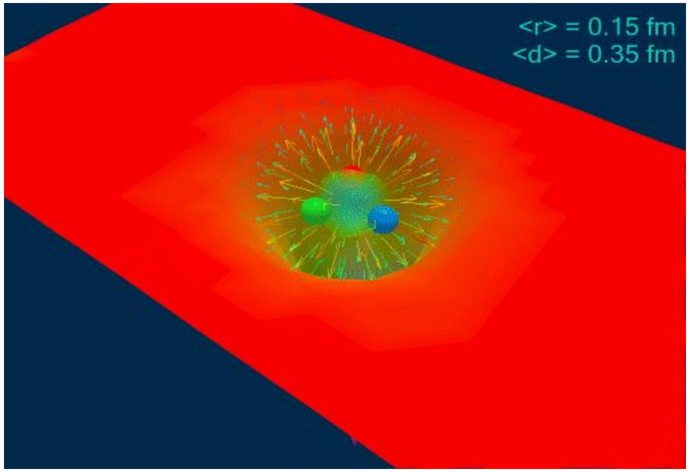
The strong force is more challenging.
It does not appear at the macroscopic level.
Only operates at distance scales $\sim$ fm
It is so strong that perturbation theory (Feynman diagrams) can't be used to predict basic phemenoma such as the formation of hadrons and confinement of quarks.

Use hadron spectra (quarkonium) to extract the form of the strong potential.

meson



baryon



Quark masses

Quark mass is a nightmare to think about. Its not an observable.

Two ways to think about it.

(1) Bare/intrinsic mass.

Rough definition:

The mass a quark would if we could measure it as a free particle - from calculations based on hadrons masses.

Obs!! $\frac{2m_{u-bare}+m_{d-bare}}{m_p} \approx \frac{2\times 4+7}{1000} \approx 0.02$ tiny !!

$\Rightarrow$ proton mass comes from quark motion and the binding between quarks.

(2) Effective/constituent mass.

Absorb interactions and motion of quarks into effective masses:

$$\frac{2m_{u-eff}+m_{d-eff}}{m_p} \approx \frac{2\times 360+360}{1000} \approx 1$$

Use definition (1)

QUARKS (Spin $\frac{1}{2}$)

Flavor	Charge	Mass (speculative)		
		Bare	Effective	
			In baryons	In mesons
d	$-\frac{1}{3}$	7.5	363	310
u	$+\frac{2}{3}$	4.2		
s	$-\frac{1}{3}$	150	538	483
c	$+\frac{2}{3}$	1100	1500	
b	$-\frac{1}{3}$	4200	4700	
t	$+\frac{2}{3}$	171000	N/A	

Quarkonium

Quarkonium - meson consisting of $q\overline{q}$

Consider $\pi^0\left(u\overline{u}, d\overline{d}\right)$ and J/ψ ($\overline{c}c$).

We want to work in non-relativistic quantum mechanics.

Which (if any) states can be treated non-relativistically ?

PARTICLE DATA

(Mass in MeV/c^2; Lifetime in Seconds; Charge in Units of Proton Charge.)

QUARKS (Spin $\frac{1}{2}$)

	Flavor	Charge	Mass (speculative)		
			Bare	Effective	
				In baryons	In mesons
First generation	d	$-\frac{1}{3}$	7.5	363	310
	u	$+\frac{2}{3}$	4.2		
Second generation	s	$-\frac{1}{3}$	150	538	483
	c	$+\frac{2}{3}$	1100	1500	
Third generation	b	$-\frac{1}{3}$	4200	4700	
	t	$+\frac{2}{3}$	>23,000		

LEPTONS (Spin $\frac{1}{2}$)

	Lepton	Charge	Mass	Lifetime	Principal decays
First generation	e	−1	0.511003	∞	—
	ν_e	0	0	∞	—
Second generation	μ	−1	105.659	2.197×10^{-6}	$e\nu_\mu\bar{\nu}_e$
	ν_μ	0	0	∞	—
Third generation	τ	−1	1784	3.3×10^{-13}	$\mu\nu_\tau\bar{\nu}_\mu$, $e\nu_\tau\bar{\nu}_e$, $\rho\nu_\tau$
	ν_τ	0	0	∞	—

MEDIATORS (Spin 1)

Mediator	Charge	Mass	Lifetime	Force	
gluon	0	0	∞	strong	
photon (γ)	0	0	∞	electromagnetic	electroweak
$W^\pm$	±1	81,800	unknown	(charged) weak	
Z^0	0	92,600	unknown	(neutral) weak	

BARYONS (Spin $\frac{1}{2}$)

Baryon	Quark content	Charge	Mass	Lifetime	Principal decays
N p	uud	+1	938.280	∞	—
N n	udd	0	939.573	900	$pe\bar{\nu}_e$
Λ	uds	0	1115.6	2.63×10^{-10}	$p\pi^-$, $n\pi^0$
Σ^+	uus	+1	1189.4	0.80×10^{-10}	$p\pi^0$, $n\pi^+$
Σ^0	uds	0	1192.5	6×10^{-20}	$\Lambda\gamma$
Σ^-	dds	−1	1197.3	1.48×10^{-10}	$n\pi^-$
Ξ^0	uss	0	1314.9	2.90×10^{-10}	$\Lambda\pi^0$
Ξ^-	dss	−1	1321.3	1.64×10^{-10}	$\Lambda\pi^-$
Λ_c^+	udc	+1	2281	2×10^{-13}	not established

BARYONS (Spin $\frac{3}{2}$)

Baryon	Quark content	Charge	Mass	Lifetime	Principal decays
Δ	uuu, uud, udd, ddd	+2, +1, 0, −1	1232	0.6×10^{-23}	$N\pi$
Σ^*	uus, uds, dds	+1, 0, −1	1385	2×10^{-23}	$\Lambda\pi$, $\Sigma\pi$
Ξ^*	uss, dss	0, −1	1533	7×10^{-23}	$\Xi\pi$
Ω^-	sss	−1	1672	0.82×10^{-10}	ΛK^-, $\Xi^0\pi^-$, $\Xi^-\pi^0$

PSEUDOSCALAR MESONS (Spin 0)

Meson	Quark content	Charge	Mass	Lifetime	Principal decays
$\pi^\pm$	$u\bar{d}$, $d\bar{u}$	+1, −1	139.569	2.60×10^{-8}	$\mu\nu_\mu$
π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	0	134.964	8.7×10^{-17}	$\gamma\gamma$
$K^\pm$	$u\bar{s}$, $s\bar{u}$	+1, −1	493.67	1.24×10^{-8}	$\mu\nu_\mu$, $\pi^\pm\pi^0$, $\pi^\pm\pi^\pm\pi^\mp$
K^0, $\bar{K}^0$	$d\bar{s}$, $s\bar{d}$	0, 0	497.72	K_S^0 0.892×10^{-10}	$\pi^+\pi^-$, $\pi^0\pi^0$
				K_L^0 5.18×10^{-8}	$\pi e\nu_e$, $\pi\mu\nu_\mu$, $\pi\pi\pi$
η	$(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$	0	548.8	7×10^{-19}	$\gamma\gamma$, $\pi^0\pi^0\pi^0$, $\pi^+\pi^-\pi^0$
η'	$(u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$	0	957.6	3×10^{-21}	$\eta\pi\pi$, $\rho^0\gamma$
$D^\pm$	$c\bar{d}$, $d\bar{c}$	+1, −1	1869	9×10^{-13}	$K\pi\pi$
D^0, $\bar{D}^0$	$c\bar{u}$, $u\bar{c}$	0, 0	1865	4×10^{-13}	$K\pi\pi$
$F^\pm$ (now $D_s^\pm$)	$c\bar{s}$, $s\bar{c}$	+1, −1	1971	3×10^{-13}	not established
$B^\pm$	$u\bar{b}$, $b\bar{u}$	+1, −1	5271	14×10^{-13}	$D + ?$
B^0, $\bar{B}^0$	$d\bar{b}$, $b\bar{d}$	0, 0	5275		
η_c	$c\bar{c}$	0	2981	6×10^{-23}	$KK\pi$, $\eta\pi\pi$, $\eta'\pi\pi$

VECTOR MESONS (Spin 1)

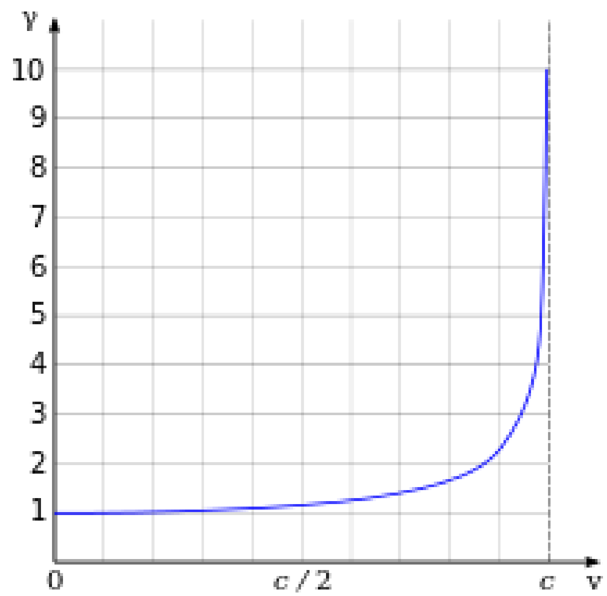
Meson	Quark content	Charge	Mass	Lifetime	Principal decays
ρ	$u\bar{d}$, $d\bar{u}$, $(u\bar{u} - d\bar{d})/\sqrt{2}$	+1, −1, 0	770	0.4×10^{-23}	$\pi\pi$
K^*	$u\bar{s}$, $s\bar{u}$, $d\bar{s}$, $s\bar{d}$	+1, −1, 0, 0	892	1×10^{-23}	$K\pi$
ω	$(u\bar{u} + d\bar{d})/\sqrt{2}$	0	783	7×10^{-23}	$\pi^+\pi^-\pi^0$, $\pi^0\gamma$
ϕ	$s\bar{s}$	0	1020	20×10^{-23}	K^+K^-, $K^0\bar{K}^0$
J/ψ	$c\bar{c}$	0	3097	1×10^{-20}	e^+e^-, $\mu^+\mu^-$, 5π, 7π
D^*	$c\bar{d}$, $d\bar{c}$, $c\bar{u}$, $u\bar{c}$	+1, −1, 0, 0	2010	$>1 \times 10^{-22}$	$D\pi$, $D\gamma$
Υ	$b\bar{b}$	0	9460	2×10^{-20}	$\tau^+\tau^-$, $\mu^+\mu^-$, e^+e^-

Hadron size: $r \sim 10^{-15}\text{m}$

Quark momentum : $p \sim \frac{\hbar}{10^{-15}} \sim \frac{10^{-34}}{10^{-15}} \sim 10^{-19}\text{kgms}^{-1}$

$p = \gamma m v, \quad m_{u,d} \sim 5 \text{ MeV}/c^2 \sim 10^{-29} \text{ kg} \;\; ; \;\; m_c \sim 1.10 \text{ GeV}/c^2 \sim 2\times 10^{-27} \text{ kg}$

$$\gamma v = \frac{p}{m} \sim \begin{bmatrix} \frac{10^{-19}}{10^{-29}} \sim 10^{10} \text{ ms}^{-1} & \left[\pi^0\left(u\bar{u}, d\bar{d}\right)\right] \\ \frac{10^{-19}}{2\times 10^{-27}} \sim 5\times 10^{7} \text{ ms}^{-1} & \left[J/\psi\left(c\bar{c}\right)\right] \end{bmatrix}$$



$\pi^0\left(u\bar{u}, d\bar{d}\right)$: $\gamma v >> \left[c = 3\times 10^8 \text{ ms}^{-1}\right] \Rightarrow$ relativistic

$: \frac{\text{quark mass}}{\pi^0 \text{ mass}} \sim \frac{5+5}{130} \sim 7\%$

$J/\psi\left(c\bar{c}\right)$: $\gamma v << \left[c = 3\times 10^8 \text{ ms}^{-1}\right] \Rightarrow$ non-relativistic

$: \frac{\text{quark mass}}{J/\psi \text{ mass}} \sim \frac{1.1+1.1}{3} \sim 70\%$

Heavy quarkonia : non-relativistic motion and smaller contribution of the strong force to heavy hadron mass.

Heavy quarkonium states

Charmonium ($c\bar{c}$) and bottomium ($b\bar{b}$) contain heavy quarks undergoing non-relativstic motion.

$\Rightarrow$ Use Schrödinger equation:

$$-\frac{1}{2\mu}\nabla^2\Psi(r)+V(r)\Psi=E\Psi$$

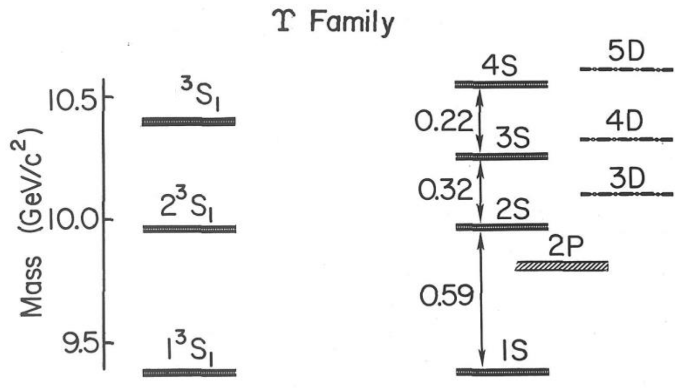
Splitting in hadrons ~ 1 GeV

Splitting in H -atom $\sim$ 10 eV

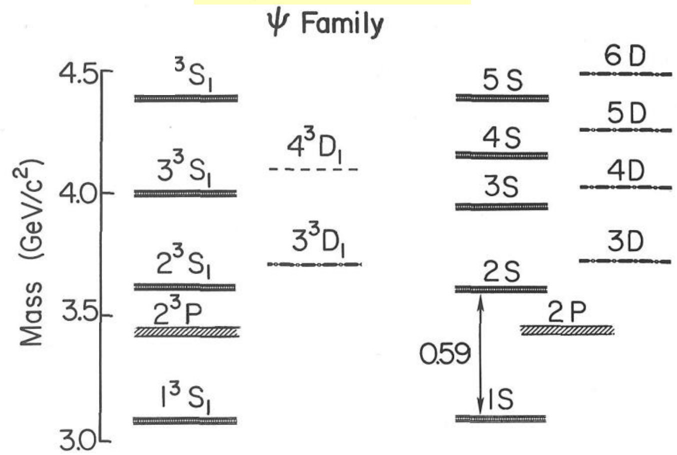
Huge difference $\sim$ 10^8

$\Rightarrow$ Strong force potential $V(r)$ can be extracted.

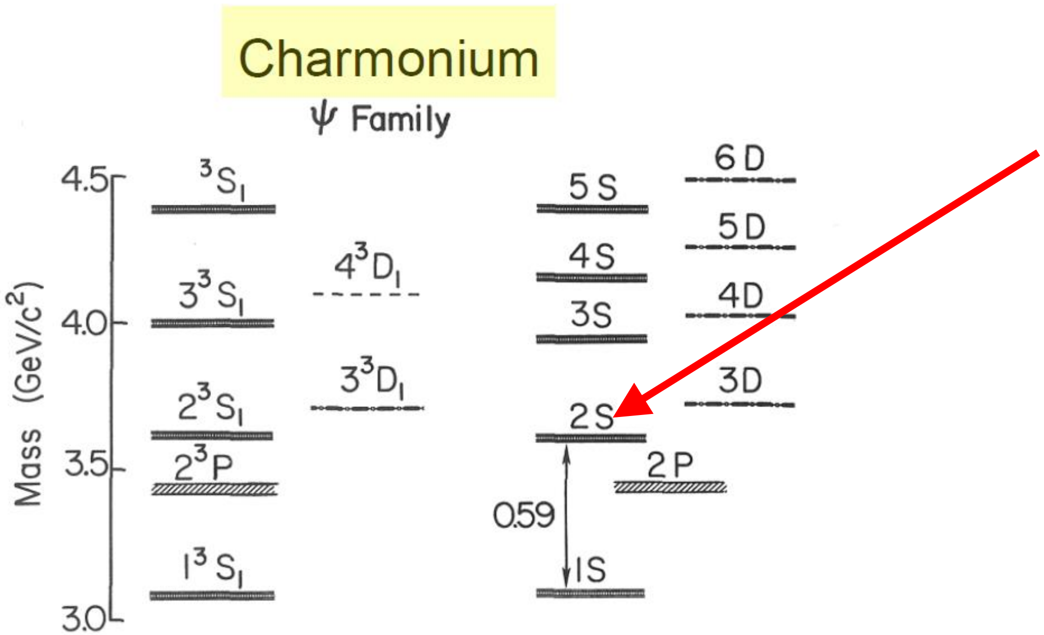
Bottomium



Charmonium

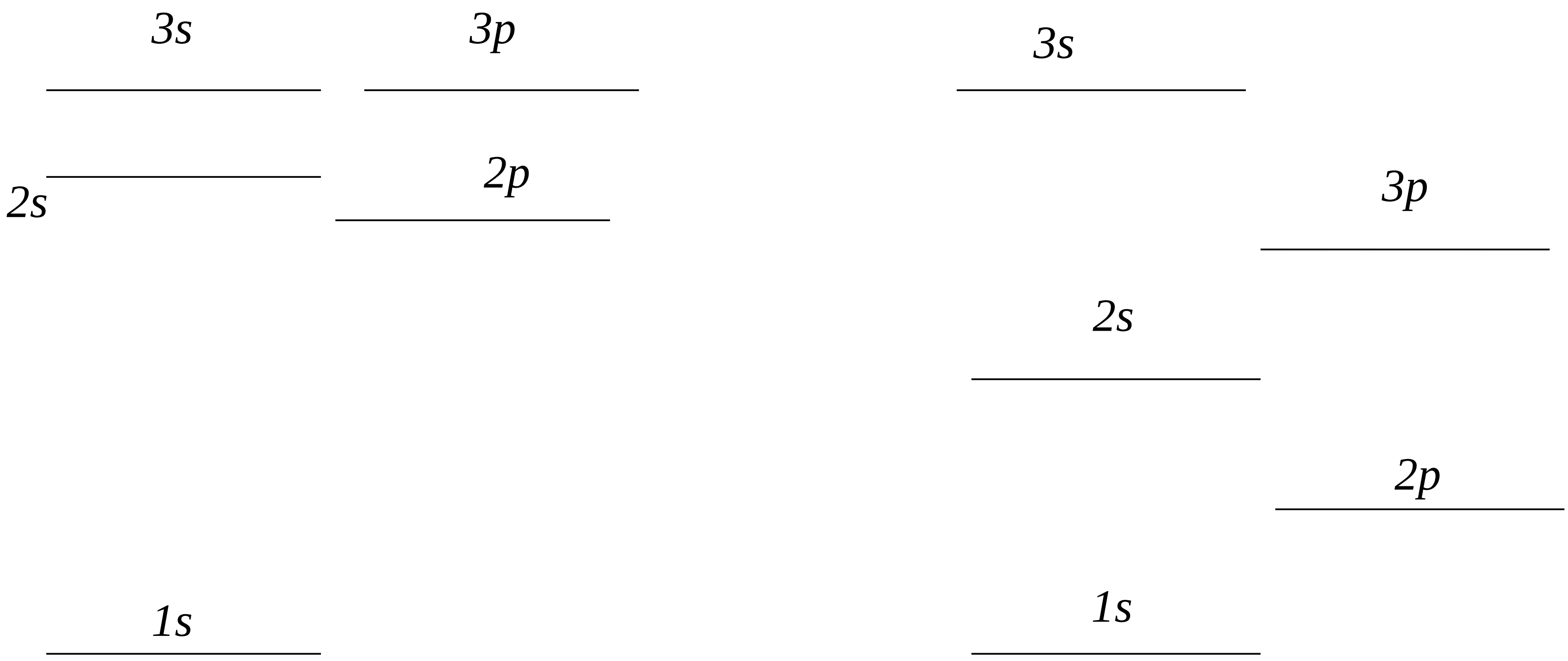


Question



Draw a Feynman diagram of a possible decay of $\psi(2S)$

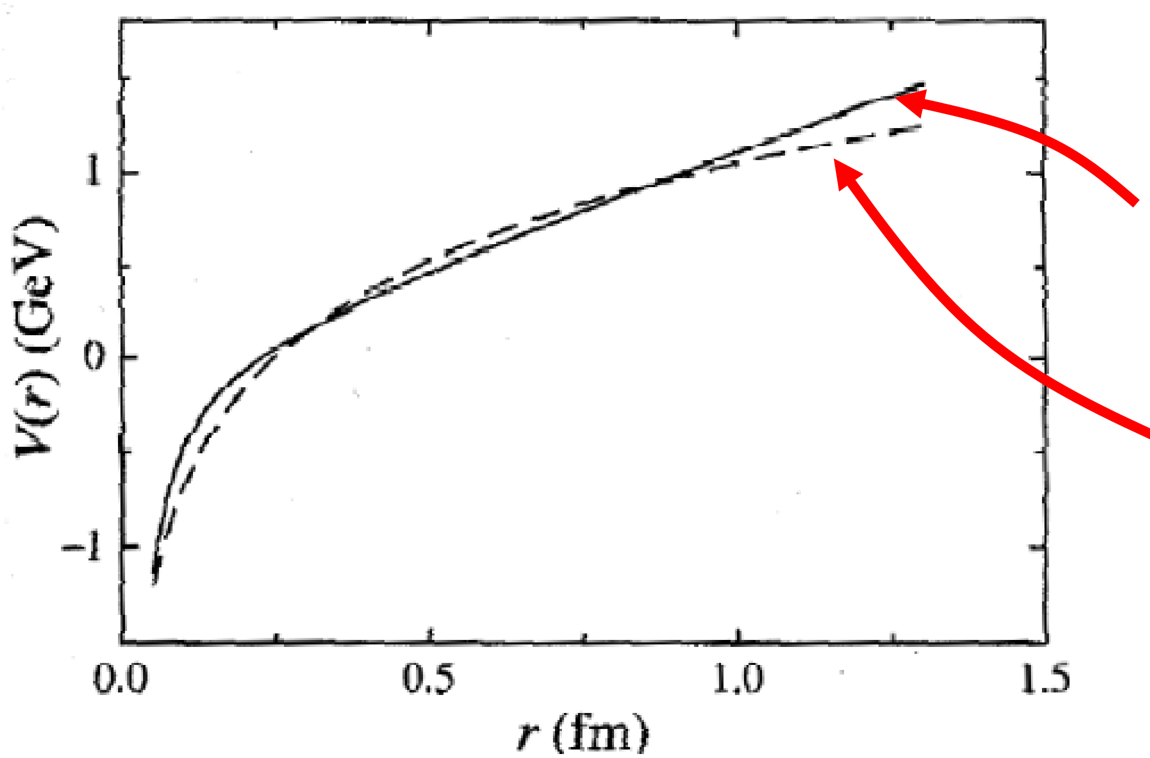
Mass states for different potentials



Coulomb $V \propto \frac{1}{r}$

Oscillator $V \propto r^2$

Potential of the strong force



Data are consistent with

$$V(r) = -\frac{a}{r} + br$$

or

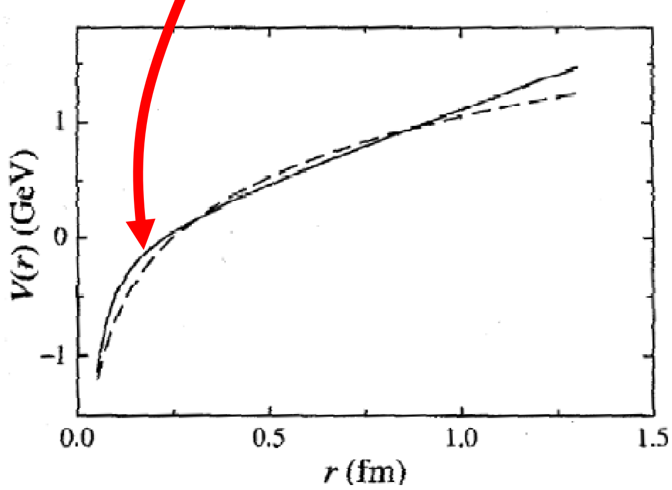
$$V(r) = a\ln(br)$$

Both curves in good agreement between 0.2 and 1fm.

Need to study the strong force over distances >1fm to get good discrimination between parameterisations. Nature doesn't grant us this!!

Question

The strong force potential $\left[V(r) = -\frac{a}{r} + br\right]$ is plotted for heavy quarkonia.



Estimate the force between a quark and anti-quark separated by a distance of 1fm.

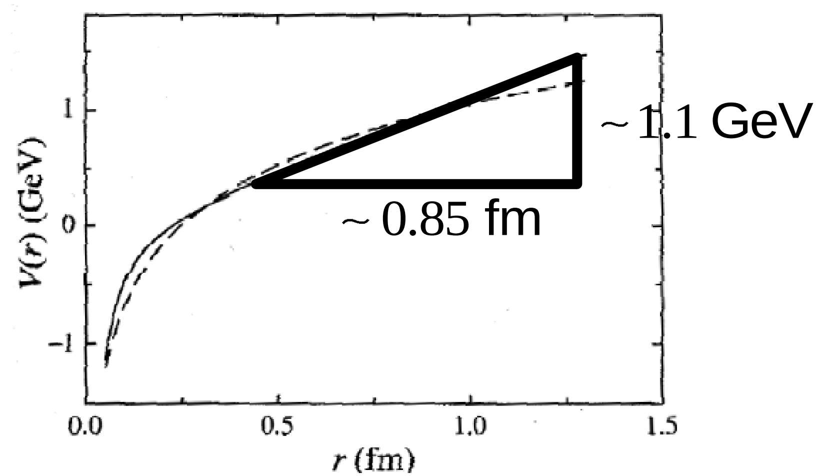
$$V(r) = -\frac{a}{r} + br$$

The term $-\frac{a}{r}$ is negligible $r > 0.5$ fm

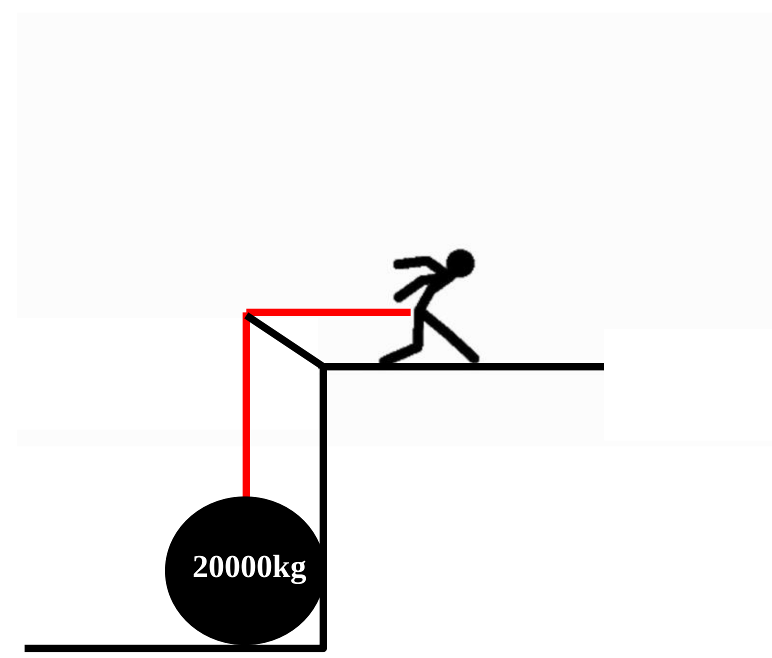
At $r = 1$ fm $V(r) \sim br$

$$\vec{F} = -\frac{\partial V}{\partial r}\hat{r} = -b\hat{r}$$

$$b \sim \frac{1.1 \times 1.602 \times 10^{-10}}{0.85 \times 10^{-15}} \sim 2 \times 10^{5} \text{ N}$$



A quark trying to escape a meson at 1fm separation must beat a force equivalent to lifting a 20 ton weight.



Confinement

- Heavy quarkonium shows that the quarks are very tightly bound in hadrons.
- This is consistent with our non-observation of free quarks.
- It does not show that free quarks can *never* be observed.
  - It would have to be shown that infinite work is needed to pull out a quark.
  - We only have information on the strong potential over a small distance (~fm)
- Understanding/proving quark confinement remains a major open question in modern physics.

Summary

- Bound states
  - Quantised energies
  - Transitions/decays between states
  - Higher momenta for constituents as size reduces
- Strong force
  - Positronium as a model
  - Use for heavy quarkonia
  - Strong force potential
- The strong force is special and strange !