Aspects of String Theory

Sayed Fawad Hassan

Theoretical Physics Group
Tata Institute of Fundamental Research
Bombay 400005, India

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Contents

SYNOPSIS

It is commonly believed that string theory is the only serious candidate for a unified theory of fundamental interactions [1]. In its framework, one can address such diverse questions as the existence of a quantum theory of gravity, the origin of fermion masses, the number of fermionic generations, restrictions on the dimensions of space-time, etc. Though a lot of insight has already been obtained, the unavailability of a proper non-perturbative formulation of the theory has prevented us from providing unique, satisfactory answers to such questions. One of the problems encountered when describing the low energy physical world in terms of string theory is the existence of a multitude of ground states associated with different compactification schemes, each giving rise to different low-energy physics. At present, there is no known mechanism by which the theory could pick up one of these ground states, presumably, leading to our observed physical universe. This state of affairs is commonly believed to be an artifact of the perturbative formulation of the theory, which is the only formulation known to us at the present. However, in the absence of a non-perturbative formulation, a first step towards a better understanding of string theory is to investigate the structure of the space of classical vacua, the so called *moduli space* of the theory. This issue acquires further importance if we remember that the low energy physics is determined not only by the choice of a vacuum state, but also by the structure of the moduli space containing that vacuum. In string theory, a classical vacuum is described by a 2-dimensional Conformal Field Theory. A study of the space of classical vacua, therefore, corresponds to studying the space of associated conformal field theories. This has been an area of intense investigation and the present work [2,3,4] is an attempt in this direction. In the following, we will dwell on these ideas in greater detail.

The basic assumption of string theory is that the elementary constituents of matter are small one-dimensional objects rather than point-like particles. The length of such a string is very small ($\sim 10^{-32}$cm) and is, therefore, observable only at very high energies. Classically this modification is a simple generalization but it gives rise to a drastically different quantum theory. In fact, on quantizing the motion of the string, one naturally obtains all the ingredients required for a description of the physical world without having to introduce them by hand as is done in theories based on point-like particles. One of the main features of string theory is that it has a conformal symmetry which, though essential for the consistency of the quantum theory, could have an anomaly at the quantum level. It is the vanishing of this anomaly which, for free strings, restricts the space-time dimensions to 26 in bosonic string theory and 10 in superstring theory. The quantum spectrum of the bosonic string contains massless states described by a rank-2 symmetric tensor (the graviton), a rank-2 antisymmetric tensor (the axion in four dimensions), and a scalar (the dilaton). The spectrum also contains an excitation of imaginary mass (the tachyon) and an infinite tower of highly massive states which are not observable at low energies. In the superstring theory, the tachyon disappears and the remaining states are paired with fermionic states, rendering the spectrum supersymmetric. The subset of massless states forms the gravitational sector of the string theory. A more realistic theory, however, is the heterotic string theory which is a hybrid between the bosonic string and the superstring theories. Besides the above states, the spectrum of this theory also contains massless gauge bosons and their superpartners belonging to the fundamental representation of an $E_8 \otimes E_8$ gauge group. This sector, at low energies, gives rise to the observed gauge bosons and chiral fermions. A string propagating in space-time can interact with a background of other strings in any of these excited states. These interactions with the background can be effectively incorporated in the theory by considering a modified vacuum where the various string modes condense, giving rise to non-zero background fields. We denote the massless bosonic background fields by the graviton field, G_{MN}, the antisymmetric tensor field, B_{MN}, the dilaton field, Φ, and the gauge field, A^a_M, where, M, N are space-time indices and a is a gauge group index. The fermionic backgrounds are set to zero. The propagation of a string in a background of its massless excitations is effectively described by a two dimensional non-linear σ-model formulated on the worldsheet swept by the string. In this model, the space-time coordinates of the string, moving in a D-dimensional space-time, appear as scalar fields on the worldsheet and the space-time dependent background fields play the role of the couplings in the theory. The condition of conformal invariance at the quantum level is implemented by setting the β-functions corresponding to these couplings to zero. At the one-loop level of the σ-model perturbation theory, this leads to a set of differential equations for the background fields which is a generalization of the Einstein's equation in general relativity. These equations can also be obtained from an action which turns out to be the same as the effective action S_{eff} of the theory obtained after integrating out the massive string modes. In the sequel, we will refer to the field equations as the *low-energy equations of motion* and to the action as the *low-energy effective action*. A configuration of the background fields which solves the low energy equations of motion describes a space-time geometry and corresponds to a classical vacuum of string theory. In this approach, the moduli space of classical vacua is determined by the space of solutions to the low energy equations of motion. Since it has not yet been possible to parametrize this space in full generality, in our study, we consider a smaller subspace corresponding to a certain class of solutions (to be specified later), for which an appropriate parametrization can be found. To put our results in a proper perspective in the following we will first give a brief account of the previous related work and then summarize our results.

In heterotic string theory the 16 extra bosonic coordinates are compactified on a torus, giving rise to the gauge degrees of freedom. Narain [5,6] considered a direct compactification of the heterotic string to $10-d$ dimensions. For constant background fields, he found that when the right moving and the left moving coordinates are separately compactified on a d-torus and a $(d+16)$-torus respectively, the actual compactification space is given by a $(2d+16)$-torus of Lorentzian signature $(d, d+16)$. The space of compactified theories is then parametrized by the group $O(d, d+16)$. More precisely, to obtain physically distinct theories, we have to consider this space modulo a subgroup Λ of $O(d, d+16)$ which generates symmetry transformations. Subsequently, various aspects of this parameter space were studied in more detail [7]. The possibility of the extension of this result to non-heterotic theories with time dependent but space independent background fields was shown by Meissner and Veneziano [8]. Considering such a theory in $d+1$ dimensions, they showed that the one-loop low energy effective action is invariant under transformations of the background fields generated by an $O(d, d)$ group. A subset of these transformations is not a symmetry of the full string theory and, therefore, relates solutions of the equations of motion which are physically distinct [9]. These deformations, thus, generate a subspace of the full moduli space of classical vacua. This construction was further extended by Sen [10] to situations where the background fields in D dimensions are independent of some d of the space-time coordinates. In [11] he related the existence of these deformations to the properties of the correlation functions in string theory and argued that they arise as a consequence of the Lorentz invariance and factorizability of the correlation functions carrying zero momentum conjugate to d of the space-time coordinates. This argument proves the existence of corrections to the $O(d,d)$ transformations to all orders in the σ-model perturbation theory. An interesting feature of these transformations is that they include, as a discrete subgroup, the usual target space duality transformations [12] which change the space-time geometry but keep the conformal field theory invariant. In our study, we first consider the relation between the $O(d,d)$ transformations on the one hand, and marginal deformations and coset construction in Wess-Zumino-Novikov-Witten (WZNW) models, on the other hand [3]. We then generalize these transformations to heterotic string theory with space-time dependent background fields and discuss an application [2]. At the end, we consider the deformations of the superstring and heterotic string theories with extended supersymmetry on the worldsheet [4]. Our results are summarized in the following.

1. $O(d,d)$ Transformations as Marginal Deformations :

As mentioned earlier, a solution of the β-function equations corresponds to a conformally invariant non-linear σ-model on the worldsheet. It is well known that if a conformal field theory has operators of conformal weight $(1,1)$ in its spectrum, then it can be perturbed to a nearby theory by adding these marginal operators to its Lagrangian. The new theory is also conformally invariant to first order in the perturbation parameter. Now, consider a conformal field theory with n marginal operators. If a perturbation by any one of these operators keeps the conformal weights of all the n operators unchanged, then the perturbations can be integrated to finite marginal deformations. Though this proves the existence of a connected family of conformal field theories, an explicit construction of this family is usually a difficult problem. In conformally invariant theories with chirally conserved currents $J^i, \bar{J}^i$ on the worldsheet, the marginal operators can simply be constructed as $\sum_{i,j} C_{ij} J^i \bar{J}^j$. When the currents commute, the marginal perturbations are integrable [13].

To relate an $O(d,d)$ transformation with an integrable marginal perturbation, we consider its action on the $SU(2)$ and the $SU(2) \otimes SU(2)$ Wess-Zumino-Novikov-Witten (WZNW) models. These models are exact conformal field theories formulated on group manifolds and have integrable marginal deformations constructed from currents in the Cartan subalgebra of the associated Kac-Moody algebra. We perform an $O(d,d)$ transformation involving the abelian isometries of the group manifold corresponding to the Cartan generators. It is then shown that such a transformation preserves the number of abelian chiral currents. Now, comparing two theories related by an infinitesimal transformation, we find that [3]

$$S_{\omega+\delta\omega} = S_\omega + \frac{1}{2\pi}\sum_{i,j=1} C_{ij}(\omega)\int d^2z J^i_\omega \bar{J}^j_\omega \tag{0.1}$$

where, ω denotes the $O(d,d)$ transformation parameters and J^i_ω, $\bar{J}^j_\omega$ are the commuting chirally and anti-chirally conserved currents of the deformed theory. This proves that $O(d,d)$ transformations generate finite marginal deformations of the above theories. We further suggested this to be extendable to all WZNW models. This was subsequently proven in the more general setting of non-linear σ-models with commuting chiral isometries by Henningson and Nappi [14]. Using their approach, and considering infinitesimal transformations, we identify the subset of the $O(d,d)$ generators which give rise to the above marginal deformations. These results clarify the meaning of the $O(d,d)$ transformations in terms of the underlying conformal field theory.

2. $O(d,d)$ Deformations and the Coset Construction :

One of the important methods of constructing new conformal field theories, starting from the ones based on Kac-Moody algebras, is the coset construction due to Goddard, Kent and Olive (GKO). When these current algebra theories are formulated as WZNW models, the GKO construction is implemented by coupling non-dynamical gauge fields to the chiral currents and integrating out the gauge fields after gauge fixing. This construction can be extended to more general non-linear σ-models with chiral currents. These models include the marginal deformations of WZNW models discussed above. With the help of this procedure, we can understand some further aspects of deformations generated by the $O(d,d)$ transformations. We first develop the gauging procedure and then apply it to our models.

Consider the non-linear σ-model action S describing the propagation of a string in a graviton and antisymmetric tensor field background (the dilaton does not appear in the action):

$$S = \frac{1}{2\pi}\int d^2z\,(G_{MN}+B_{MN})\bar{\partial}X^M\partial X^N \tag{0.2}$$

Due to the presence of B_{MN}, which can be written as Wess-Zumino term, the isometries in S cannot be gauged unambiguously by a covariant derivative replacement. However, to preserve conformal invariance, in models of the type considered by us, it is sufficient to couple gauge fields to chirally conserved currents of the theory. For theories with $2r$ commuting chiral isometries, and with backgrounds in the block-diagonal form, we use the results of [15,16] to give a prescription for gauging by a covariant derivative replacement [3]. Let ζ^M_{La} and ζ^M_{Ra} be the Killing vectors generating independent left and right chiral isometries and let $\zeta^M_a = \zeta^M_{La} + \zeta^M_{Ra}$ generate the isometries to be gauged. To maintain conformal invariance, we first add a total derivative term of the form $\int d^2z\, \partial_{[M}w_{N]}\bar{\partial}X^M\partial X^N$ to the ungauged action such that the Nöther currents become equal to the chiral currents. Here,

$$2\partial_{[M}w_{N]}\zeta^N_a = g_{MN}(\zeta^N_{Ra} - \zeta^N_{La}) - b_{MN}(\zeta^N_{La} + \zeta^N_{Ra}) \tag{0.3}$$

Then, with $D_\alpha X^M = \partial_\alpha X^M - \zeta^M_a A^a_\alpha$ as the covariant derivative, the gauged action is given by

$$S^{gauged} = \frac{1}{2\pi}\int d^2z\,(G_{MN} + B_{MN} + 2\partial_{[M}w_{N]})\bar{D}X^M DX^N \tag{0.4}$$

Having developed the gauging procedure, we use it to understand some features of marginal deformations generated by the $O(d,d)$ group in terms of the coset construction. One important observation is the fact that in the case of $SU(2)$ (or the $SL(2,R)$) WZNW model, the axial $U(1)$ and the vector $U(1)$ cosets, plus a free boson, appear as the limit points on the deformation line generated by marginal deformations of the original theory. To explain this, we gauge the marginally deformed $SU(2)$ theory and show that the outcome is independent of the deformations parameter. In particular, we can choose a value for the parameter such that a free boson decouples from the rest of the theory. Now, the gauging procedure will eliminate this free boson keeping the rest of the theory unchanged. It is now clear that this remaining part cannot be anything but the coset model [3]. This leads to an understanding of the more general observation made in [17] that, for theories with chiral currents, certain fixed $O(d,d)$ transformations produce the coset models plus free bosons.

Next, we consider the $O(3,3)$ deformations of the $(SU(2)/U(1)) \otimes SU(2)$ model which give rise to a three parameter family of theories. Since one of the isometries in this model does not lead to a chirally conserved current, all the deformations are not of the $J\bar{J}$-type. We can, however, understand this in terms of the deformations of the parent $SU(2) \otimes SU(2)$ theory. But first, let us formulate the problem in a more general setup. It was shown in [18] that any theory in D dimensions, with d commuting isometries, can be obtained as a coset of a theory in $D + d$ dimensions with d chiral and d anti-chiral commuting isometries. The non-trivial $O(d,d)$ transformations of the D-dimensional theory then correspond to gauging different linear combinations of the chiral currents in the $(D+d)$-dimensional theory. We refer to this as *twisted* *gauging*. However, if we consider a coset with respect to some $n < d$ of the chiral currents, then the resulting $(D+d-n)$-dimensional theory will have a larger class of $O(2d-n, 2d-n)$ deformations though a smaller number of them are accounted for by the twisted gaugings of the original model. The case we have considered is an example of this : Gauging two $U(1)$ currents in the $SU(2)\otimes SU(2)$ theory gives a 1-parameter family of theories with $D=4$ which is related to the $O(2,2)$ deformations of the $(SU(2)/U(1))\otimes(SU(2)/U(1))$ theory. However, gauging a single $U(1)$ leads to the $(SU(2)/U(1))\otimes SU(2)$ model with $D=5$ and a 3-parameter family of deformations all of which are not accounted for by the twisted gauging of the parent theory or the $J\bar{J}$-type deformations of the coset theory. To understand the origin of these deformations in relation to the parent theory, we analyse the twisted gauging of the marginal deformations of the $SU(2)\otimes SU(2)$ theory. By judiciously selecting the twisting and marginal deformation parameters in this theory, we show that all the deformations of the $SU(2)/U(1)\otimes SU(2)$ model are reproduced [3].

3. The $O(d, d+p)$ Transformations in Heterotic String Theory :

In heterotic string theory, 16 of the left-moving bosonic coordinates are compactified on a torus giving rise to an $E_8\otimes E_8$ gauge group. The momenta corresponding to these compact coordinates are the root vectors of this group, which, in the Cartan-Weyl basis, appear as structure constants in the commutators involving elements from the Cartan subalgebra and the rest of the $E_8\otimes E_8$ algebra. Now, in analogy with our discussion of superstring theory above, we consider correlation functions which are neutral under p of the Cartan generators of the gauge group and which also carry zero momentum along d space directions. We then show that these correlation functions are invariant under the action of an $O(d)\otimes O(d+p)$ group which corresponds to independent Lorentz rotations of the d right moving and $d+p$ left moving coordinates. If the d coordinates include the time direction, then the group is modified to $O(d-1,1)\otimes O(d+p-1,1)$. Clearly, the $O(d+p)$ factor transforms the neutral and the charged states into each other. This hints at the existence of a similar transformation for some restricted configurations of the background fields which keep the heterotic string low-energy effective action invariant. In terms of the background fields, the restrictions mentioned above imply that G_{MN}, B_{MN}, Φ and A^a_M are independent of d coordinates and the gauge fields A^a_M belong to a configuration that commutes with p of the Cartan generators of the gauge group. Before proceeding further, let us introduce some notation: We label the d

coordinates on which the background fields do not depend by $m, n, \ldots$ and the remaining ones by $\mu, \nu, \ldots$. Also, the label R is used to denote p of the $U(1)$ generators with which the gauge fields commute. The non-zero gauge fields which lie outside the Cartan subalgebra are labeled by the gauge group index α. Furthermore, for the sake of simplicity, we consider backgrounds such that $G_{m\mu} = B_{m\mu} = 0$ and where the only non-vanishing components of the gauge field are A^α_μ and A^R_m. Now, let $\hat{G}$ and $\hat{B}$ denote the $d \times d$ matrices G_{mn} and B_{mn} respectively, and let $\hat{A}$ denote the $p \times d$ matrix A^R_m. Using these we construct a $(2d+p) \times (2d+p)$-dimensional matrix M [2] given by

$$M = \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1}K^T & -\hat{G}^{-1}\hat{A}^T \\ K\hat{G}^{-1} & K\hat{G}^{-1}K^T & -K\hat{G}^{-1}\hat{A}^T \\ -\hat{A}\hat{G}^{-1} & -\hat{A}\hat{G}^{-1}K^T & \hat{A}\hat{G}^{-1}\hat{A}^T \end{pmatrix} \tag{0.5}$$

where, $K = \hat{G} + \hat{B} + \frac{1}{2}\hat{A}^T\hat{A}$. η is the flat Euclidean metric if X^m are all space coordinates, and the flat Lorentzian metric of signature $(1, d-1)$ if they include the time coordinate as well. We then show that the low-energy effective action for the heterotic string, which is written as

$$\begin{aligned} S_{eff} = -\int d^D x \sqrt{\det G} e^{-\Phi} \Big(\Lambda - G^{MN} \partial_M \Phi \partial_N \Phi - R^{(D)}(G) \\ + \tfrac{1}{12} H_{MNP} H^{MNP} + \tfrac{1}{4} \textstyle\sum_a F^a_{MN} F^{aMN} \Big), \end{aligned} \tag{0.6}$$

is invariant under a transformation of the background fields given by

$$\begin{aligned} \widetilde{M} &= \Omega M \Omega^T \\ \widetilde{\Phi} &= \Phi + \frac{1}{2} ln\,(\det\widetilde{G}/\det G) \end{aligned} \tag{0.7}$$

Here, H_{MNK} and F^a_{MN} are the field strengths for B_{MN} and the gauge field background A^a_M, respectively, and Ω is an element of $O(d, d+p)$. This larger group also contains general linear coordinate transformations and constant gauge transformations of B_{mn} and A^R_m. For general background fields which are not of block diagonal form, we prove the existence of the transformation when all gauge fields are abelian. In this case, the transformation is still given by (0.7) where M is now a $(2D+16)\times(2D+16)$ matrix given by (0.5) with $\hat{G}, \hat{B}$ and $\hat{A}$ replaced by the full matrices G, B and A. Also, the $O(d, d+16)$ group element Ω is now embedded in the larger group $O(D, D+16)$.

Since the $O(d+p)$ subgroup mixes the space-time and gauge group directions, we can use the transformations to construct charged solutions of the equations of motion starting from neutral solutions of the theory [2]. As an application we consider a magnetically charged six-brane solution of the low energy heterotic theory in ten dimensions which carries zero electric and antisymmetric tensor field charges. The transformed solution is characterized by two parameters which come from the $O(1,1)\otimes O(2,1)$ group and correspond to the electric and antisymmetric tensor field charges of the new solution. Note that the non-compact nature of the transformations is because of the inclusion of time as one of the isometry directions. The space-time structure of the new solution differs from that of the old one and consists of a singularity surrounded by two event-horizons.

To summarize, we prove the existence of an $O(d, d+p)$ group acting on the space of conformally invariant backgrounds in heterotic string theory. The existence of this transformation is guaranteed to all orders in the σ-model perturbation theory and can be used to generate charged solutions starting from neutral ones.

4. The $O(d,d)$ and $O(d,d+16)$ Deformations and Extended Worldsheet Supersymmetry :

It is well known that the N=1 worldsheet supersymmetry in string theory, can be extended to N=2 or N=4 without changing the field content of the theory [19,20]. This is achieved by requiring the background fields to satisfy certain restrictions which turn out to be related to the existence of some well defined geometrical structures of the target space manifold. These extended supersymmetries are of great importance because of the restrictions which they impose on the structure of the moduli space of classical vacua as well as their natural relation to space-time supersymmetry. Since the deformations discussed above change the space-time geometry in a non-trivial way, it is important to study how they affect the structures associated with extended supersymmetry on the worldsheet. To answer this question, we first consider the superstring and then the heterotic string theories with extended supersymmetry on the worldsheet and show that the transformations discussed above preserve the extended supersymmetry. They, therefore, generate superconformal deformations though a superfield formulation of the theory, in terms of chiral and twisted chiral superfields, is not in general preserved. In the following, we present our results in greater detail.

In superstring theory the transformation of background fields under the $O(d,d)$ group is given by equations (0.7) where M is given by equation (0.5) with the gauge field background set to zero, and where $\hat{G}$ and $\hat{B}$ are replaced by the full $D\times D$ matrices G and B.. The group element Ω then has to be embedded in the larger group $O(D,D)$. To analyse the deformations of theories with extended supersymmetry, we find it convenient to rewrite these transformations in terms of a $2D\times D$ matrix ξ given by [4]

$$\xi = \begin{pmatrix} e \\ K\,e \end{pmatrix}, \qquad \text{with} \quad G^{-1} = e\,\eta\,e^T \tag{0.8}$$

The matrix ξ is related to M by $M = \xi\eta\xi^T$ and under the action of the $O(d,d)$ group transforms to $\widetilde{\xi} = \Omega\xi$. From this, we can obtain the transformation of e as $\widetilde{e} = Q_- e$, where, the matrix Q_- is determined by (0.8) and the explicit form of the transformation matrix Ω. Note that though the $O(d,d)$ transformation is global, the matrix Q_- which implements the transformation on e depends on space-time through its dependence on the background fields. Due to this fact, if we have a vector bundle the sections of which transform as $\widetilde{V} = Q_- V$, then, on the bundle we can introduce an $O(d,d)$ induced connection ω_μ^- given by

$$\begin{aligned} \omega_\mu^- &= \tfrac{1}{2}\, e\eta\xi^T L\partial_\mu \xi e^{-1} - e\partial_\mu e^{-1}, \\ \widetilde{\omega}_\mu^- &= Q_-\,\omega_\mu^- Q_-^{-1} - \partial_\mu Q_- Q_-^{-1} \end{aligned} \tag{0.9}$$

where, the second line gives the transformation properties of the induced connection under a deformation.

The superstring theory has a left-right symmetry and is invariant under the worldsheet parity transformation provided we also transform $B \to -B$. Now, the backgrounds can be deformed either before or after a parity transformation. Correspondingly, we can either work with the pair of quantities (Q_-, ω_μ^-) or with their parity transformed counterparts, (Q_+, ω_μ^+). These considerations prove very useful while studying the deformations of theories with extended worldsheet supersymmetry. It is well known that the N=1 supersymmetry can be extended to an N=2 supersymmetry if the target manifold admits two integrable complex structures, J^- and J^+ which are covariantly constant with respect to the torsionful connections on the σ-model manifold, and which give rise to two hermitian structures on the manifold. Under an $O(d,d)$ transformation, these complex structures transform as

$$
\begin{aligned}
\tilde{J}^- &= Q_- J^- Q_-^{-1} \\
\tilde{J}^+ &= Q_+ J^+ Q_+^{-1}
\end{aligned}
\tag{0.10}
$$

Using the structures introduced above, we show that the $O(d,d)$ deformations preserve all of the conditions of extended (2,2) supersymmetry on the worldsheet [4]. We then generalize the proof to any extended (p,q) supersymmetry in superstring theory. By studying the transformations of the complex structures, we show that generic deformations do not preserve the known representations of the theory in terms of chiral and twisted chiral superfields. We also write down the transformations of the complex structures under the duality transformations which form a discrete subgroup of the $O(d,d)$ group. Our approach clarifies the relation between these duality transformations and a duality which exists in the N=2 models [19].

The case of extended supersymmetry in heterotic string theory can be treated along similar lines [4]. The transformations (0.7) can still be written as $\tilde{\xi} = \Omega\,\xi$ where the $(2D+16)\times(D)$ dimensional matrix ξ is now given by

$$
\xi = \begin{pmatrix} e \\ K\,e \\ -A\,e \end{pmatrix}
\tag{0.11}
$$

Using this, we can again construct the the matrix M as $M = \xi\eta\xi^T$ and write down an induced connection as in (0.9). With the help of these constructions, we show that $O(d,d+16)$ deformations preserve the (0,2) and (0,4) extended worldsheet supersymmetry in heterotic string theory.

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Chapter 1

INTRODUCTION TO O(d, d) TRANSFORMATIONS IN STRING THEORY

1.1 INTRODUCTION

By now it is commonly believed that the standard model of particle physics which combines the electro-weak theory with quantum chromodynamics is a valid description of particle interactions only at low energies. Though it may be possible to find a more fundamental theory which extends the range of validity of the standard model to somewhat higher energies, it is clear that a theory of fundamental interactions which is valid at all energy scales, should also accommodate gravity. This is because at very high energies or, equivalently, at very short distances, gravity becomes important and the quantum gravity effects are no longer negligible. A quantum theory of gravity, however, has not yet been formulated and, therefore, a naive attempt to unify gravity with other interactions is not successful. String theory [1, 2], on the other hand, tries to circumvent this problem in a very novel way. Here, the notion of a point-like particle as the fundamental object is replaced by that of a small one dimensional object or a string of linear dimension $\sim 10^{-32} cm$. The quantum spectrum of the theory then contains a massless spin 2 state identified as the graviton. A phenomenologically more viable version of the theory is the heterotic string theory. This theory also contains gauge bosons and can give rise to chiral fermions in 4 dimensions. This version of string theory is rich enough to be a candidate for a unified theory of fundamental interactions. In order to put the discussion in the thesis in a proper perspective, in this section we will briefly review some relevant aspects of string theory.

Similar to point-like mechanics, the dynamics of a string is governed by an action which is the area of the worldsheet it sweeps while moving in space-time. Formulated in terms of the worldsheet action, the theory of a string propagating in a D dimensional space-time is equivalent to the theory of D massless scalar fields in two dimensions with conformal symmetry. The absence of negative norm states from the physical spectrum of the string requires the conformal symmetry to be non-anomalous at the quantum level. This imposes certain restrictions on the allowed set of parameters of the theory. For the simplest case of strings in flat space-time, this requires $D = 26$ for the bosonic string and $D = 10$ for the superstring and the heterotic string theories. In order to obtain a description of the real world as observed at low energies, one has to compactify the extra spatial dimensions in the theory. In the framework of first quantized string theory, this amounts to choosing a proper vacuum for the theory. In general the vacuum can get modified by the formation of condensates acting as background fields. The propagation of a string in a background of the string massless excitations is described by a non-linear σ-model in two dimensions where the background fields appear as the σ-model couplings. The bosonic massless background fields in the bosonic string and superstring theories are the metric G_{MN}, the antisymmetric tensor field B_{MN}, and the dilaton Φ. The heterotic string theory also contains a gauge field background A^a_M. Here, M, N are space-time indices, and a is a gauge group index. The condition of absence of the conformal anomaly in the presence of background fields implies that the β-functions corresponding to these couplings should vanish. To one-loop level in the σ-model perturbation theory, this gives rise to a set of differential equations for the background fields[3, 4, 5, 6]. These generalize the field equations in general relativity to the case when gravity is coupled to an antisymmetric tensor field, a dilaton field and, in case of heterotic string theory, to a gauge field. The β-function conditions can also be obtained as equations of motion from an action which generalizes the Einstein-Hilbert action. This is the one-loop low-energy effective action in string theory and, for the non heterotic case and for zero fermionic backgrounds, is given by

$$S = -\int d^D x \sqrt{-\det G} e^{-\Phi} \left(\Lambda - R^{(D)}(G) - G^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{12} H_{MNP} H^{MNP} \right) \tag{1.1}$$

Here, $R^{(D)}$ denotes the D dimensional Ricci scalar and the 3-form H_{MNP} is the field strength associated with the antisymmetric tensor field: $H_{MNP} = \partial_M B_{NP} + cyclic\ permutations$. Λ

is the cosmological constant equal to $(D-26)/2$ for the bosonic string and $(D-10)/2$ for the superstring. Higher loop corrections to the β-functions result in higher derivative terms in the effective action. However, as we shall see later, there are some aspects of the theory for which it is sufficient to consider only the one-loop effective action and then prove the existence of corrections to all orders in the σ-model perturbation theory.

A solution to the β-function equations corresponds to a classical ground state of string theory. Since such a solution is not unique, the perturbative ground state in the theory is degenerate. The theory does not seem to have any preference for any one of these ground states and, therefore, in order to carry out the quantization, one has to select one of these vacuum states arbitrarily. Theories based on different vacua give rise to different space-time physics. The arbitrariness in the selection of the ground state is, at present, one of the fundamental problems of string theory. It is believed that the string non-perturbative effects may provide a solution to this problem, though no such non-perturbative effect has yet been identified. Before making an attempt to address the problem of vacuum degeneracy in the framework of the perturbative formulation of string theory, it is logical to explore the structure of the space of classical vacua. This study is also important from another point of view since it turns out that a lot of information about the space-time physics, in a specific ground state, is contained in the structure of the space of the classical vacua containing that ground state. An important progress in understanding some aspects of the moduli space classical vacua was Narain's classification of the space of toroidal compactifications of the heterotic string theory [7, 8]. In this work it was found that it is possible to compactify the bosonic sector of the heterotic string on a $d+16$ dimensional lattice and the fermionic sector on a d dimensional lattice giving rise to a theory in $10-d$ dimensions. However, the full compactification lattice is an integer, even, self-dual, Lorentzian lattice of signature $(d, d+16)$. This lattice can be rotated by the group $O(d, d+16)$ of which a subgroup$O(d) \otimes O(d+16)$ is a symmetry of the physical spectrum. The elements belonging to the coset $O(d, d+16)/(O(d) \otimes O(d+16))$, on the other hand, deform the compactification lattice and give rise to new compactified theories. These transformations, therefore, generate the moduli space of compactifications. An equivalent way of understanding these deformations is to interpret them as transformations of the constant background fields in the theory. For these kinds of backgrounds, the β-functions vanish trivially. For a comprehensive list of references on the subsequent work in this field see [9]. Another important feature of string theory is the existence of a target space duality symmetry [20]-[22] which relates different target space background field configurations giving rise to the same conformal field theory. Such different target space geometries are therefore physically indistinguishable. In order to obtain the moduli space of inequivalent classical vacua, one has to consider the space of vacua modulo the duality transformations.

A very important class of backgrounds corresponds to non-trivial, *i.e.*, space-time dependent solutions of the β-function equations. In the past few years a large number of such solutions with rich space-time structures have been constructed [10]. Some of these solutions have also been obtained as exact conformal field theories. An important problem is to understand the analogue of Narain's group for such solutions. Progress in this direction was first made by Meissner and Veneziano [11],[12]-[15]. They found that for string backgrounds in $d+1$ dimensions which are independent of d coordinates, the space of classical vacua is invariant under the action of an $O(d,d)$ group of transformations. The d coordinates in this case need not necessarily be compact. These transformations were generalized by Sen [16, 17] to backgrounds in D dimensions which are independent of d coordinates and their string theory origin was clarified. The relation between these transformations and the coset construction was considered in [18, 19]. The present thesis is devoted to a further investigation of these transformations and is based on the work appearing in [23, 24, 25, 26]. To prepare the ground for a presentation of this work, in the next section we will introduce the $O(d,d)$ transformations following references [16] and [17]. In section 3, we will outline the content of the rest of the thesis.

1.2 O(d, d) TRANSFORMATIONS IN STRING THEORY

Consider the bosonic massless background fields G_{MN}, B_{MN} and Φ of the bosonic string or the superstring theories in D dimensions which are independent of d of the space coordinates $\hat{X}^m$; $1 \leq m \leq d$. We denote the remaining coordinates on which the solution may depend, and which for the time being also include the time direction, by $\tilde{X}^\mu$; $1 \leq \mu \leq D-d$. The dynamics of these fields is governed by the action (1.1) when restricted to the above class of backgrounds. For simplicity, let us further restrict ourselves to background fields G and B which are block diagonal in form;

$$G = \begin{pmatrix} \tilde{G}_{\mu\nu} & 0 \\ 0 & \hat{G}_{mn} \end{pmatrix} \quad , \quad B = \begin{pmatrix} \tilde{B}_{\mu\nu} & 0 \\ 0 & \hat{B}_{mn} \end{pmatrix} \tag{1.2}$$

For the above decomposition of fields and after integrating by parts, the low-energy effective action (1.1) takes the form

$$\begin{aligned} S = -\int d^d\hat{X} \int d^{D-d}\tilde{X} \sqrt{-\det \tilde{G}} e^{-\chi} \Big(-\frac{1}{8} \tilde{G}^{\mu\nu} Tr(\tilde{\partial}_\mu M L \tilde{\partial}_\nu M L) \\ +\Lambda - \tilde{G}^{\mu\nu} \tilde{\partial}_\mu \chi \tilde{\partial}_\nu \chi - \tilde{R}^{(D-d)}(\tilde{G}) + \frac{1}{12} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} \Big) \end{aligned} \tag{1.3}$$

where,

$$L = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} , \tag{1.4}$$

$$\chi = \Phi - \ln \sqrt{\det \hat{G}} \tag{1.5}$$

and M is a $2d \times 2d$ matrix defined as

$$M = \begin{pmatrix} \hat{G}^{-1} & -\hat{G}^{-1}\hat{B} \\ \hat{B}\hat{G}^{-1} & \hat{G} - \hat{B}\hat{G}^{-1}\hat{B} \end{pmatrix} \tag{1.6}$$

It is clear that the action (1.3) is invariant under transformations of the type

$$M \to M' = \Omega M \Omega^T \tag{1.7}$$

$$\Phi \to \Phi' = \Phi + \ln \sqrt{\det G' / \det G} \tag{1.8}$$

$$\tilde{G}_{\mu\nu} \to \tilde{G}_{\mu\nu}, \qquad \tilde{B}_{\mu\nu} \to \tilde{B}_{\mu\nu} \tag{1.9}$$

provided Ω keeps the metric L invariant:

$$\Omega L \Omega^T = L \tag{1.10}$$

This condition implies that the transformations Ω form an $O(d,d)$ group.

We may consider three different classes of $O(d,d)$ transformations, denoted by Ω_1, Ω_2, and Ω_3. This division is made depending on the action of these classes on the background fields. These three classes are parametrized as follows:

$$\Omega_1 = \begin{pmatrix} (A^T)^{-1} & 0 \\ 0 & A \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \tag{1.11}$$

and

$$\Omega_3 = \frac{1}{2} \begin{pmatrix} \mathcal{S}+\mathcal{R} & \mathcal{R}-\mathcal{S} \\ \mathcal{R}-\mathcal{S} & \mathcal{S}+\mathcal{R} \end{pmatrix} \tag{1.12}$$

Here, A is a general $d \times d$ matrix, C is an antisymmetric $d \times d$ matrix, and $\mathcal{S}$ and $\mathcal{R}$ are both $O(d)$ matrices. The elements Ω_1 generate transformations of the form $\hat{G} \to A\hat{G}A^T$, $\hat{B} \to A\hat{B}A^T$ on the backgrounds which can also be obtained by a $GL(d)$ transformation of the coordinates $\hat{X}^m$ as: $\hat{X}^m \to A^m_n \hat{X}^n$. The elements Ω_2 generate gauge transformations of B of the form $\hat{B}_{mn} \to \hat{B}_{mn} + C_{mn}$ corresponding to a gauge parameter $\Lambda_m = C_{mn}\hat{X}^n$. These linear coordinate transformations and gauge transformations are symmetries of the theory when the coordinates $\hat{X}^m$ are non-compact. For compact coordinates they give rise to inequivalent backgrounds. The genuinely non-trivial transformations are generated by the elements Ω_3 which parametrize an $O(d) \otimes O(d)$ subgroup of $O(d,d)$. To be more precise, we have to consider the coset $(O(d) \otimes O(d))/O(d)$ which is the set of elements Ω_3 modulo the diagonal $O(d)$ subgroup given by $\mathcal{S} = \mathcal{R}$. This diagonal $O(d)$ gives rise to d dimensional rotations which have already been included in Ω_1. The elements Ω_3 keep the trivial backgrounds $\hat{G}_{mn} = \delta_{mn}, \hat{B}_{mn} = 0$ invariant. When the d coordinates $\hat{X}^m$ on which the backgrounds do not depend also include the time direction (say $\hat{X}^1$), then the matrices $\mathcal{S}$ and $\mathcal{R}$ are elements of $O(d-1,1)$ defined by

$$\mathcal{S}\eta\mathcal{S}^T = \eta, \qquad \mathcal{R}\eta\mathcal{R}^T = \eta \tag{1.13}$$

where, $\eta = diag(-1,1,\ldots,1)$. The matrices Ω_3 now parametrize an $O(d-1,1) \otimes O(d-1,1)$ subgroup of $O(d,d)$ and are given by

$$\Omega_3 = \frac{1}{2} \begin{pmatrix} \eta(\mathcal{S}+\mathcal{R})\eta & \eta(\mathcal{R}-\mathcal{S}) \\ (\mathcal{R}-\mathcal{S})\eta & \mathcal{S}+\mathcal{R} \end{pmatrix} \tag{1.14}$$

Note that all the above discussion is in the context of the low-energy effective action and is, therefore, accurate to one-loop level in the σ-model perturbation theory. However, in [16] the origin of these transformations was traced back to the factorization properties, in a 2-dimensional conformal field theory, of the correlation functions which carry zero momentum along d of the coordinates. We will discuss this in detail in section 3.2 in the context of heterotic string theory. This association with an underlying conformal field theory guarantees the existence of the transformations to all orders in the σ-model perturbation theory though the explicit form of the transformation will receive higher order corrections [27, 28, 29].

In our study in this thesis, we shall be mainly interested in transformations generated by the elements Ω_3 of the $O(d,d)$ group given in (1.12) or (1.14), as the case may be. These transformations (modulo the diagonal subgroup given by $\mathcal{S} = \mathcal{R}$) are symmetries of the restricted action (1.3) but not of the full action (1.1). They, therefore, relate background field configurations which are physically distinct and thus generate, at least, a subspace of the full moduli space of the theory. The moduli space of truly inequivalent vacua is obtained after identifying geometrically distinct background field configurations which are related by duality transformations. These transformations form a discrete subgroup of $O(d) \otimes O(d)$.

1.3 OUTLINE OF THE THESIS

In this section we briefly outline the structure of the present thesis which studies some aspects of the $O(d,d)$ transformations of the classical vacua in string theory and its generalization to $O(d,d+p)$ transformations in heterotic string theory. More detailed information about the content of each chapter can be found in the introduction to that chapter. The result of the investigation in each chapter is summarized at the end of that chapter.

The remaining three chapters are organized as follows. In Chapter 2, we consider the connection between $O(d,d)$ transformations and marginal deformations. As an example of theories in which the integrable marginal operators are easily identifiable, we consider the $SU(2)$ and the $SU(2) \otimes SU(2)$ Wess-Zumino-Novikov-Witten (WZNW) models explicitly. For the coset models (which are equivalent to gauged WZNW models) where the marginal operators are not easily identifiable, we relate the $O(d,d)$ transformations of the coset to the marginal deformations of the ungauged theory. We also clarify some relations observed between the cosets and certain fixed $O(d,d)$ transformations of the $SU(2)$ WZNW theory. In Chapter 3, we consider the heterotic string theory and show that, after properly restricting the background fields, the $O(d,d)$ group is generalized to an $O(d,d+p)$ group of transformations. These transformations can relate backgrounds with zero gauge field to ones with non-trivial gauge fields. As an example, we start with a black six-brane solution with only a magnetic charge and obtain a solution with electric, magnetic and antisymmetric tensor gauge field charges. In Chapter 4, we write the $O(d,d)$ and $O(d,d+16)$ transformations in terms of the target space vielbeins and show that they preserve the $N=2$ and $N=4$ extended supersymmetries on the worldsheet. This is provided the complex structures associated with these supersymmetries are also independent of the d coordinates which are isometries of the backgrounds and with respect to which the transformation is performed. We also obtain, explicitly, the transformation properties of the complex structures associated with the extended worldsheet supersymmetries.

Chapter 2

THE O(d, d) TRANSFORMATIONS, MARGINAL DEFORMATIONS AND THE COSET CONSTRUCTION

2.1 INTRODUCTION

One of the important objects of study in string theory is the space of classical vacua or, equivalently, the space of conformal field theories corresponding to these classical vacua. The notion of a space of conformal field theories can be introduced in an intuitive way, in the context of statistical mechanical systems in two dimensions at a critical point. At this point, the correlation length diverges and the theory becomes conformally invariant. The behaviour of the system is then characterized by the properties of a fixed point in the parameter space. At the fixed point, the system is described by a conformal field theory with its central charge given by the Zamolodchikov's c-function [30]. It is, however, well known that some systems have, as their fixed points, a continuous subspace of the parameter space [31] on which the c-function remains constant. The dimensionality of this subspace of fixed points, obviously, depends on the number of marginal operators in the theory. In 2-dimensional conformal field theories, these are operators of conformal weight $(1, 1)$ with which the theory can be perturbed [31, 32]. The integrability of these perturbations imposes some conditions on the marginal operators. Consider a theory with n marginal operators. The condition of mutual integrability then demands that a perturbation by any one of these operators should keep the conformal weights of all the n operators unchanged. These conditions can be formulated as properties of the operator algebra of the theory.

In chapter 1, we introduced an $O(d,d)$ group of transformations which act on the massless background fields in string theory which have translational invariance along d of the space-time coordinates. It was mentioned that a subset of these transformations relate physically distinct background field configurations while preserving the conformal invariance of the non-linear σ-model which describes the propagation of the string in these background fields. For related work also see [55]-[59] and [60]. It is expected that such transformations of the background fields should be related to the marginal deformations of the underlying conformal field theory. Unfortunately, for a generic conformally invariant non-linear σ-model, the operator content of the theory and its operator algebra are not easily identifiable. This prevents us from finding the explicit relation between $O(d,d)$ transformations and marginal deformations in these theories. To overcome this problem, we consider a class of non-linear σ-models, namely, the Wess-Zumino-Novikov-Witten (WZNW) models, for which the conformal field theory structure is well known [33, 34, 35, 36]. Since the analysis of this chapter is mainly in the context of these models, in the following, we will make a small digression to review some of their relevant features.

A WZNW model is essentially a non-linear σ model with a group manifold G as its target space. It is defined by the action

$$S(g) = \frac{k}{8\pi}\int_\Sigma d^2x\, Tr(\partial_\alpha g^{-1}\partial^\alpha g) + \frac{k}{12\pi}\int_B d^3x\, \epsilon^{ijk}\, Tr(\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g g^{-1}) \tag{2.1}$$

where, g is a map from the worldsheet Σ to the group manifold G, and B is a solid ball in three dimensions with Σ as its boundary. The integral over B is a Wess-Zumino term and, in a given parametrization of g, can be written as a surface integral over Σ. For $g, h \in G$ one can verify that

$$S(gh) = S(g) + S(h) - \frac{k}{4\pi}\int d^2x\,(\eta^{\alpha\beta} + \epsilon^{\alpha\beta}) Tr(g^{-1}\partial_\alpha g \partial_\beta h h^{-1}) \tag{2.2}$$

The above relation is known as the Polyakov-Wiegman identity [38]. Using this identity, it is easy to see that the action (2.1) is invariant under holomorphic and anti-holomorphic transformations of the form $g(z,\bar z) \to \bar\Omega(\bar z)\, g(z,\bar z)\, \Omega(z)$, where, $z = exp(ix^+)$, $\bar z = exp(ix^-)$,$x^\pm = x^0 \pm x^1$ and $\Omega\,, \bar\Omega \in G$. The chirally conserved currents associated with this symmetry are

$$J^a(z) = kTr(T^a g^{-1}\partial_z g)\,, \quad \bar J^a(\bar z) = kTr(T^a \partial_{\bar z} g g^{-1}) \tag{2.3}$$

where, T^a are the generators of the group G. These currents generate a Kac-Moody algebra of central charge k based on the Lie algebra of G. The energy-momentum tensor is now given in terms of the chiral currents by the Sugawara construction as

$$T(z) = \frac{1}{2k + c_G} \sum_{a=1}^{dimG} : J^a(z)\, J^a(z) : \tag{2.4}$$

with a similar expression for $\bar{T}(\bar{z})$. Here, c_G is the quadratic Casimir of the adjoint representation of G. The energy-momentum tensor defined above generates a Virasoro algebra of central charge $c(G,k) = 2k\mathrm{dim}G/(2k + c_G)$. Let the group G contain a subgroup H and let T^G and T^H denote the energy-momentum tensors associated with G and H, respectively. As shown in [43], we can construct a new energy-momentum tensor $T^{G/H} = T^G - T^H$ which satisfies the Virasoro algebra with central charge $c(G/H,k) = c(G,k) - c(H,k)$. This is the GKO coset construction. It was shown in [45]-[50] that, in the Lagrangian formulation, the GKO construction is implemented by gauging a WZNW model. This means coupling appropriate chiral currents to non-dynamical gauge fields and integrating out the gauge fields after fixing the gauge [51, 54].

The WZNW models have some easily identifiable marginal deformations. From the OPE's of the currents and the energy-momentum tensor it follows that $J^a(z)$ and $\bar{J}^a(\bar{z})$ have conformal dimensions $(1,0)$ and $(0,1)$, respectively. One can therefore construct bilinears in these currents of conformal dimension $(1,1)$. Though these operators are marginal, they are not all integrable. However, from the analysis of [39] it follows that any bilinear combination of the form $\sum_{a,b} C_{ab} J^a \bar{J}^b$, which contains currents only from the Cartan subalgebra of the group G, is an integrable marginal operator. On the other hand, choosing the Euler parametrization (see equation (2.34) below) for the group element g and using the Polyakov-Wiegman identity (2.2), one can see that the background fields in the σ-model (2.1) are invariant under translations generated by the elements from the Cartan subalgebra. These backgrounds, therefore, admit deformations generated by the $O(2r,2r)$ group where r is the rank of the group G.

The above digression shows that WZNW models and their cosets are ideal for examining the relation between $O(d,d)$ transformations and integrable marginal deformations. For the coset models, where the integrable marginal perturbations are not as easily identifiable, one may examine the conformal field theory origin of $O(d,d)$ transformations with reference to the parent theory from which they are obtained by gauging. In the remaining sections of this chapter, we will address these issues by explicitly considering the $SU(2)$ and the $SU(2) \otimes SU(2)$ WZNW models and the $SU(2)/U(1)$ and $SU(2) \otimes (SU(2)/U(1))$ cosets. The chapter is organized as follows: In section 2, we prove the equivalence between $O(2,2)$ transformations and the marginal deformations in the $SU(2)$ WZNW models. In section 3, this result is extended to $O(4,4)$ transformations of the $SU(2) \otimes SU(2)$ model and the general case is discussed. In section 4, we construct the $O(3,3)$ deformations of the $SU(2) \otimes (SU(2)/U(1))$ coset model. To understand the conformal field theory origin of the deformations of this coset, in section 5 we discuss the gauging of non-linear σ-models with chiral currents. Using these results, in section 6 we construct some $U(1)$ cosets of our deformed WZNW models. We show that the axial and vector $U(1)$ cosets of the $SU(2)$ theory, when augmented with a free boson, are limit points on the deformation line of the $SU(2)$ theory. We also discuss the origin of the deformations of the $SU(2) \otimes (SU(2)/U(1))$ coset model. The conclusions of this chapter are summarized in section 7.

The content of this chapter is entirely based on the work appearing in reference [24] and partly elaborated upon in reference [26].

2.2 O(2,2) TRANSFORMATIONS AND MARGINAL DEFORMATIONS IN THE SU(2) WZNW MODEL

To explore the connection between $O(d,d)$ transformations and marginal perturbations in WZNW models, we first consider an $SU(2)_k$ WZNW model defined by the action (2.1) where g is now an $SU(2)$ group element and the constant k specifies the level of the associated Kac-Moody algebra. This model describes a conformal field theory of central charge $c = \frac{3k}{k+2}$ and gives the well known *2-d* black hole [52] on gauging [51]. As discussed in the previous section, this model has chiral isometries which are generated by the holomorphic and anti-holomorphic currents of the associated Kac-Moody algebra. In terms of the coordinates on the group manifold, the chiral isometries generated by the Cartan subalgebra are manifest if we choose the Euler parametrization for the $SU(2)$ group elements. In this parametrization,

g is given by

$$g = e^{i\theta_L\sigma_2/2}e^{i\phi\sigma_1/2}e^{i\theta_R\sigma_2/2} \tag{2.5}$$

where, θ_L, θ_R and ϕ are the coordinates on the $SU(2)$ group manifold. Now, using the Polyakov-Wiegman identity (2.2), the action (2.1) can be written as

$$S(\phi, \theta_L, \theta_R) = \frac{k}{2\pi}\int d^2z(\bar{\partial}\phi\partial\phi + \bar{\partial}\theta_L\partial\theta_L + \bar{\partial}\theta_R\partial\theta_R + 2\cos\phi\bar{\partial}\theta_L\partial\theta_R) \tag{2.6}$$

Note that the $SU(2)$ model is related to the $SL(2,\mathrm{R})$ model by the replacements $\phi \to ir$ and $k \to -k$. Therefore, though we will explicitly consider the $SU(2)$ model, the conclusions are trivially extendible to the $SL(2,\mathrm{R})$ model. The above action is in the form of a non-linear σ-model from which we can read off the metric and the antisymmetric tensor fields. Due to the conformal symmetry of the model, these background field configurations automatically satisfy the zero β-function conditions. The action (2.6) has three chiral and three anti-chiral isometries. However, in the parametrization chosen, only one isometry in each chiral sector acts as a translation :

$$\begin{aligned} &\delta\theta_L = v(z)\,, \qquad \delta\theta_R = 0 \\ &\bar{\delta}\theta_L = 0\,, \qquad \bar{\delta}\theta_R = \bar{v}(\bar{z}) \end{aligned} \tag{2.7}$$

This gives rise to the following conserved chiral currents

$$\begin{aligned} J &= \frac{1}{2}k(\partial\theta_L + \cos\phi\,\partial\theta_R) \\ \bar{J} &= \frac{1}{2}k(\bar{\partial}\theta_R + \cos\phi\,\bar{\partial}\theta_L) \end{aligned} \tag{2.8}$$

Note that in the action (2.6), the background fields are independent of the two coordinates θ_L and θ_R. The model, therefore, has a one-parameter deformation generated by $O(2,2)$ transformations. Our aim in this section is to identify this one-parameter family as the $J\bar{J}$-type deformations of the $SU(2)_k$ model. However, before that, we find it more convenient to work with the new coordinates

$$\theta = \frac{1}{2}(\theta_L - \theta_R)\,, \qquad \widetilde{\theta} = \frac{1}{2}(\theta_L + \theta_R) \tag{2.9}$$

in terms of which the metric G_{MN} is diagonal. After shifting the B-field by a constant matrix, which can also be implemented by an $O(2,2)$ transformation, the action (2.6) takes the form

$$S(\phi,\theta,\tilde{\theta}) = \frac{k}{2\pi}\int d^2z\Big(\frac{1}{4}\bar{\partial}\phi\partial\phi + \sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta + \cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\partial\tilde{\theta} + \cos^2\frac{\phi}{2}(\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta)\Big) \quad (2.10)$$

From this, we can read off the σ-model couplings as

$$G = \begin{pmatrix} k/4 & 0 & 0 \\ 0 & k\sin^2\frac{\phi}{2} & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \\ 0 & -k\cos^2\frac{\phi}{2} & 0 \end{pmatrix} \quad (2.11)$$

The dilaton field can be obtained using the above background fields in the β-function equations and one gets $\Phi = 0$. Also in terms of the new coordinates, the chiral transformations (2.7) become

$$\begin{aligned} \delta\theta &= \frac{1}{2}v(z)\ , & \delta\tilde{\theta} &= \frac{1}{2}v(z) \\ \bar{\delta}\theta &= -\frac{1}{2}\bar{v}(\bar{z})\ , & \bar{\delta}\tilde{\theta} &= \frac{1}{2}\bar{v}(\bar{z}) \end{aligned} \quad (2.12)$$

and the conserved currents (2.8) take the form

$$\begin{aligned} J &= k\,(\,\sin^2\frac{\phi}{2}\partial\theta + \cos^2\frac{\phi}{2}\partial\tilde{\theta}\,) \\ \bar{J} &= k\,(\,-\sin^2\frac{\phi}{2}\bar{\partial}\theta + \cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\,) \end{aligned} \quad (2.13)$$

As discussed in the previous section, in a conformal field theory based on a Kac-Moody algebra, all marginal operators of the form $\sum_{i,j} C_{ij}J_i\bar{J}_j$, with J_i and $\bar{J}_i$ from the Cartan subalgebra, are integrable [39]. Therefore, a small deformation of the $SU(2)$ WZNW model can be obtained by adding a marginal perturbation of the form

$$O = \frac{\delta\lambda}{2\pi}\int d^2z\, J\bar{J} \quad (2.14)$$

to the action (2.6). The integrability of this perturbation implies the existence of a continuous family of conformal field theories parametrized by λ, such that two theories corresponding to two adjacent values of λ are related by a generalization of the operator (2.14). However, except for some relatively simple cases, like the theory of a free boson compactified on a circle, the form of the deformed action $S_{(\lambda)}$ cannot be obtained in a straightforward way. In the following, we show that for WZNW models a solution to this problem is provided by $O(d,d)$ transformations.

Before considering the $O(2,2)$ transformations of the $SU(2)$ WZNW model (2.10), we should emphasize an intricacy related to the compactness of the coordinates. This will prove useful for our later analysis. When the coordinates are compact, a new set of backgrounds obtained by a coordinate transformation is not equivalent to the old backgrounds if we assume the same periodicity for the new and the old coordinates. In fact, since these coordinates are angular variables, the new background will generically correspond to a singular metric even if the original metric was non-singular. A trivial example of this is the metric $dr^2 + r^2 d\theta^2$ in polar coordinates. If we define $\theta' = \theta/C$, and consider the case where θ' is an angular variable with period 2π, then the new metric $dr^2 + C^2 d(\theta')^2$ has a conical singularity at $r = 0$. Coming to our model (2.10), since θ and $\widetilde{\theta}$ have periodicities of 2π, the metric does not have coordinate singularities at $\phi = 0$ and $\phi = \pi$. However, as we shall see, the field configuration obtained by a non-trivial $O(d,d)$ transformation of a WZNW theory is quite often singular. It, therefore, has to be followed by a general coordinate transformation in order to give a non-singular metric.

Now we are in a position to study the relation between the $O(2,2)$ transformations and the marginal deformations of the $SU(2)_k$ WZNW model. Consider a transformation of the background fields G_{MN}, B_{MN} and Φ given in(2.11) by the elements Ω_3 of the $O(2,2)$ group as given in (1.12). For $\mathcal{S}$ and $\mathcal{R}$, we choose the parametrization

$$\mathcal{R} = \mathcal{S}^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{2.15}$$

The new metric, anti-symmetric tensor field and dilaton are obtained from (1.7) and (1.8). It turns out that the new metric has conical singularities at $\phi = 0$ and π which are removed by the scalings $\theta \to \theta/(\cos\alpha - k\sin\alpha)$ and $\widetilde{\theta} \to \widetilde{\theta}/\cos\alpha$. This restores the periodicities of θ and $\widetilde{\theta}$ to 2π. We also make a further transformation $B_{\tilde{\theta}\theta} \to B_{\tilde{\theta}\theta} + L$, where L is a constant given by

$$L = \cos\alpha(k\cos\alpha + \sin\alpha) \tag{2.16}$$

This corresponds to locally adding a total derivative term to the action and does not change the β-function of the theory. The above choice of L is motivated by our objective of interpreting $O(2,2)$ transformations as marginal deformations. Moreover, this choice is not unique and we will refer to this point while discussing the gauging of the $O(2,2)$ transformed model. The new σ-model action obtained after these transformations, which are all contained in the $O(2,2)$ group, is given by

$$\begin{aligned} S_{(\alpha)}(\phi,\theta,\tilde{\theta}) &= \frac{k}{2\pi}\int d^2z\Big(\frac{1}{4}\bar{\partial}\phi\partial\phi + \frac{1}{\Delta}(\cos\alpha - k\sin\alpha)^2\sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta \\ &+ \frac{1}{\Delta}\cos^2\alpha\cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\partial\tilde{\theta} - \frac{1}{\Delta}\cos^2\alpha\sin^2\frac{\phi}{2}(\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta)\Big) \end{aligned} \tag{2.17}$$

with a dilaton field Φ given by

$$\Phi = -\ln\Delta \tag{2.18}$$

where,

$$\Delta = \cos^2\alpha + k\,(k\sin^2\alpha - 2\sin\alpha\cos\alpha)\cos^2\frac{\phi}{2} \tag{2.19}$$

Note that there is a freedom of adding an overall constant to the expression for Φ which we have not displayed explicitly. We will often use this freedom while comparing different solutions. As discussed in the previous chapter, although the explicit form of the transformed action (2.17) is valid only in the large k limit, the existence of corrections to it is guaranteed to all orders in $1/k$ [16, 23, 19].

The action (2.17), along with the dilaton (2.18), describes a one-parameter family of conformal field theories parametrized by α, with the $SU(2)_k$ WZNW model given by $\alpha = 0$. For small α the change in the action (2.10) under the $O(2,2)$ transformation is given by

$$\delta S \equiv S_{(\alpha)} - S_{(0)} \simeq \frac{\alpha}{\pi}\int d^2z\,J\bar{J} \tag{2.20}$$

where, J and $\bar{J}$ are the currents (2.13). A comparison with (2.14) shows that an infinitesimal $O(2,2)$ transformation generates a marginal deformation of the $SU(2)_k$ theory. To generalize this result to finite α, we notice that the action in (2.17) is invariant under the chiral transformations

$$\begin{aligned} \delta\theta &= \frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2}v(z)\,, & \delta\tilde{\theta} &= \frac{1}{2}\frac{1}{\cos^2\alpha}v(z) \\ \bar{\delta}\theta &= -\frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2}\bar{v}(\bar{z}) & \bar{\delta}\tilde{\theta} &= \frac{1}{2}\frac{1}{\cos^2\alpha}\bar{v}(\bar{z}) \end{aligned} \tag{2.21}$$

which are generalizations of (2.12). The corresponding conserved chiral currents have the form

$$
\begin{aligned}
J_{(\alpha)} &= \frac{k}{\Delta}(\sin^2\frac{\phi}{2}\partial\theta + \cos^2\frac{\phi}{2}\partial\widetilde{\theta}) = J/(\Delta) \\
\bar{J}_{(\alpha)} &= \frac{k}{\Delta}(-\sin^2\frac{\phi}{2}\bar{\partial}\theta + \cos^2\frac{\phi}{2}\bar{\partial}\widetilde{\theta}) = \bar{J}/(\Delta)
\end{aligned}
\tag{2.22}
$$

Now, in terms of $J_{(\alpha)}$ and $\bar{J}_{(\alpha)}$, a small $O(2,2)$ transformation of $S_{(\alpha)}(\phi,\theta,\widetilde{\theta})$ can be calculated to be

$$
S_{(\alpha+\delta\alpha)} = S_{(\alpha)} + \frac{\delta\alpha}{\pi}\cos\alpha(\cos\alpha - k\sin\alpha)\int d^2z\, J_{(\alpha)}\bar{J}_{(\alpha)} \tag{2.23}
$$

The above equation shows that appropriate $O(2,2)$ transformations of the $SU(2)_k$ WZNW model generate a continuous line of conformal field theories which are related by marginal perturbations. The precise relationship is as follows. The $O(2,2)$ transformation involves two of the target space coordinates, θ and $\widetilde{\theta}$, on which the background fields do not depend. Global shifts in these coordinates are therefore commuting isometries of the backgrounds, giving rise to conserved isometry currents. If this symmetry is extendible to local shifts with only holomorphic or anti-holomorphic dependences on the worldsheet coordinates, as in (2.21), then the theory contains a pair of chiral currents, and hence, a marginal operator of the form (2.14). The integrability of this marginal perturbation, to all orders in α, follows from the analysis of [39] which is performed at the $SU(2)$ point, where, the conformal structure of the theory is known. Equation (2.23) then shows that perturbations by this operator are reproduced by appropriate $O(2,2)$ transformations involving θ and $\widetilde{\theta}$. The integrability insures that the new theory $S_{(\alpha+\delta\alpha)}$ also contains chiral currents and can be perturbed further.

Thus, we have proved that an infinite series of marginal perturbations of the type $J\bar{J}$ can effectively be summed up by a finite $O(2,2)$ transformation. This transformation also gives the expression for the new dilaton field obtained after the deformation. Some features of the deformation line of the $SU(2)_k$ model will be discussed in section 6.

2.3 MARGINAL DEFORMATIONS OF THE SU(2) $\otimes$ SU(2) WZNW MODEL

Now, as a step to generalize the result of the previous section, we investigate the relation between $O(d,d)$ transformations and current-current marginal perturbations in the next simplest WZNW theory, the $SU(2)_{k_1} \otimes SU(2)_{k_2}$ model. This model is defined by the action

$$S = S_1(\,\phi, \theta, \tilde{\theta}\,) + S_2(\,r, t, \tilde{t}\,) \tag{2.24}$$

Here S_1 is the action given in equation (2.10) with $k = k_1$ and S_2 is obtained from S_1 by the replacements $(k_1, \phi, \theta, \tilde{\theta}\,) \to (k_2, r, t, \tilde{t}\,)$. The backgrounds in S are independent of the four coordinates $\theta, \tilde{\theta}, t, \tilde{t}$ and can be transformed by an $O(4,4)$ group of transformations. As stated before, a general $O(4,4)$ transformation may be parametrized by an $O(4)$ transformation followed by a gauge transformation of the anti-symmetric tensor field and a general linear coordinate transformation. It can be verified that for the models of the type considered here, only a 4-parameter family of elements from $O(4)$ will generate non-trivial deformations.

To study the connection between $O(4,4)$ transformations and marginal deformations in the $SU(2) \otimes SU(2)$ model, it is sufficient to consider the action of various $O(2)$ subgroups of $O(4)$ individually. Note that $O(4)$ contains a subgroup $O(2) \otimes O(2)$ which transforms each one of the $SU(2)$ theories separately. The action of this subgroup is a trivial extension of the $O(2,2)$ transformation of the $SU(2)$ model considered in the previous section and the σ-model action of the transformed theory is given by,

$$S_{(\alpha,\beta)} = S_{1\,(\alpha)}(\,\phi, \theta, \tilde{\theta}\,) + S_{2\,(\beta)}(\,r, t, \tilde{t}\,) \tag{2.25}$$

where, each term on the right hand side is defined by (2.17). As an example of the remaining $O(2)$ subgroups, we consider the one which twists the θ and t coordinates and corresponds to parametrizing the elements Ω_3, given in (1.12), of the $O(4,4)$ group by choosing

$$\mathcal{R} = \mathcal{S}^{-1} = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\gamma & 0 & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

The transformed backgrounds are obtained from equations (1.6), (1.7) and (1.8). The new metric has conical singularities at $\phi = 0$ and $r = 0$ which are removed by scalings $\theta \to \theta / \cos\gamma$

and $t \to t/\cos\gamma$, restoring the periodicities of $\theta, \tilde{\theta}, t$ and $\tilde{t}$ to 2π. After performing a gauge transformation on the antisymmetric tensor field B by a constant matrix, $B_{\theta t} \to B_{\theta t} - \sin\gamma\cos\gamma$, the transformed σ-model action and the dilaton field are given by

$$
\begin{aligned}
S_{(\gamma)} &= \frac{1}{2\pi}\int d^2z \Big[\frac{k_1}{4}\bar{\partial}\phi\partial\phi + \frac{k_2}{4}\bar{\partial}r\partial r \\
&+ \frac{1}{\Delta}\Big(k_1\cos^2\gamma\sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta + k_2\cos^2\gamma\sin^2\frac{r}{2}\bar{\partial}t\partial t \\
&+ k_1(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\partial\tilde{\theta} \\
&+ k_2(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\cos^2\frac{r}{2}\bar{\partial}\tilde{t}\partial\tilde{t} \\
&+ k_1\cos^2\gamma\cos^2\frac{\phi}{2}(\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta) + k_2\cos^2\gamma\cos^2\frac{r}{2}(\bar{\partial}t\partial\tilde{t} - \bar{\partial}\tilde{t}\partial t) \\
&- k_1k_2\sin\gamma\cos\gamma\sin^2\frac{\phi}{2}\sin^2\frac{r}{2}(\bar{\partial}\theta\partial t - \bar{\partial}t\partial\theta) \\
&- k_1k_2\sin\gamma\cos\gamma\sin^2\frac{\phi}{2}\cos^2\frac{r}{2}(\bar{\partial}\theta\partial\tilde{t} + \bar{\partial}\tilde{t}\partial\theta) \\
&+ k_1k_2\sin\gamma\cos\gamma\cos^2\frac{\phi}{2}\sin^2\frac{r}{2}(\bar{\partial}t\partial\tilde{\theta} + \bar{\partial}\tilde{\theta}\partial t) \\
&+ k_1k_2\sin\gamma\cos\gamma\cos^2\frac{\phi}{2}\cos^2\frac{r}{2}(\bar{\partial}\tilde{\theta}\partial\tilde{t} - \bar{\partial}\tilde{t}\partial\tilde{\theta})\Big)\Big] \\
\Phi &= -\ln\Delta
\end{aligned}
\tag{2.27}
$$

where,

$$
\Delta = \cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2}\sin^2\frac{r}{2} \tag{2.28}
$$

Now, for an infinitesimal $O(2,2)$ transformation, the change in the action (2.24) can be easily calculated from (2.27) and is given by

$$
\delta S = \frac{-\gamma}{2\pi}\int d^2z\,\Big(J_1\bar{J}_2 - J_2\bar{J}_1\Big) \tag{2.29}
$$

where, $J_1, \bar{J}_1$ and $J_2, \bar{J}_2$ are the $U(1)$ chiral currents in the $SU(2)_{k_1}$ and $SU(2)_{k_2}$ theories in (2.24), respectively. This, clearly, is an integrable marginal perturbation of the untransformed action S given in (2.24). To identify the theory given by the action $S_{(\gamma)}$, for finite γ, as the marginal deformation of the theory with $\gamma = 0$, we note that the action (2.27) is invariant under the following chiral transformations of the coordinates:

$$
\begin{aligned}
&\delta\theta = \frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}v(z)\ ,\ \delta\tilde{\theta} = \frac{1}{2}v(z)\ ,\ \delta t = 0\ ,\ \delta\tilde{t} = \frac{1}{2}k_1\tan\gamma\, v(z)\\
&\bar{\delta}\theta = -\frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}\bar{v}(\bar{z})\ ,\ \bar{\delta}\tilde{\theta} = \frac{1}{2}\bar{v}(\bar{z})\ ,\ \bar{\delta} t = 0\ ,\ \bar{\delta}\tilde{t} = -\frac{1}{2}k_1\tan\gamma\,\bar{v}(\bar{z})\\
&\delta\theta = 0\ ,\ \delta\tilde{\theta} = -\frac{1}{2}k_2\tan\gamma\, u(z)\ ,\ \delta t = \frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}u(z)\ ,\ \delta\tilde{t} = \frac{1}{2}u(z)\\
&\bar{\delta}\theta = 0\ ,\ \bar{\delta}\tilde{\theta} = \frac{1}{2}k_2\tan\gamma\,\bar{u}(\bar{z})\ ,\ \bar{\delta} t = -\frac{\cos^2\gamma + k_1k_2\sin^2\gamma}{2\cos^2\gamma}\bar{u}(\bar{z})\ ,\ \bar{\delta}\tilde{t} = \frac{1}{2}\bar{u}(\bar{z})
\end{aligned}
\tag{2.30}
$$

These invariances give rise to the following conserved chiral currents

$$
\begin{aligned}
J_{1\,(\gamma)} &= \frac{1}{\Delta}[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\,J_1 + \sin\gamma\cos\gamma k_1\cos^2\frac{\phi}{2}J_2\,]\\
\bar{J}_{1\,(\gamma)} &= \frac{1}{\Delta}[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{r}{2})\,\bar{J}_1 - \sin\gamma\cos\gamma k_1\cos^2\frac{\phi}{2}\bar{J}_2\,]\\
J_{2\,(\gamma)} &= \frac{1}{\Delta}[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\,J_2 - \sin\gamma\cos\gamma k_2\cos^2\frac{r}{2}J_1\,]\\
\bar{J}_{2\,(\gamma)} &= \frac{1}{\Delta}[(\cos^2\gamma + k_1k_2\sin^2\gamma\sin^2\frac{\phi}{2})\,\bar{J}_2 + \sin\gamma\cos\gamma k_2\cos^2\frac{r}{2}\bar{J}_1\,]
\end{aligned}
\tag{2.31}
$$

In terms of these currents, the variation of $S_{(\gamma)}$ under a small $O(2,2)$ transformation is given by

$$
\begin{aligned}
S_{(\gamma+\delta\gamma)} = S_{(\gamma)} \quad &+ \quad \frac{\delta\gamma}{2\pi(\cos^2\gamma + k_1k_2\sin^2\gamma)^2}\int d^2z\Big(2\sin\gamma\cos\gamma(k_2J_{1\,(\gamma)}\bar{J}_{1\,(\gamma)} + k_1J_{2\,(\gamma)}\bar{J}_{2\,(\gamma)})\\
&- \quad (\cos^2\gamma - k_1k_2\sin^2\gamma)(J_{1\,(\gamma)}\bar{J}_{2\,(\gamma)} - J_{2\,(\gamma)}\bar{J}_{1\,(\gamma)})\Big)
\end{aligned}
\tag{2.32}
$$

This shows that in the continuous one-parameter family of conformal field theories described by the action $S_{(\gamma)}$, two adjacent theories are related by a marginal perturbation constructed from the product of a holomorphic and an anti-holomorphic current.

The action (2.27) was obtained by an $O(2,2)$ transformation which twists the coordinates θ and t. The full set of non-trivial transformations also contains three other subgroups which mix the two $SU(2)$ theories in (2.24). The corresponding transformed actions can be directly obtained from (2.27) by the following replacements:

$$\begin{aligned} &(\theta - \tilde{t}) \text{ twisting : } \; r \to r + \pi,\, t \leftrightarrow \tilde{t} \\ &(\tilde{\theta} - t) \text{ twisting : } \; \phi \to \phi + \pi,\, \theta \leftrightarrow \tilde{\theta} \end{aligned} \tag{2.33}$$

By combining the above two replacements, one obtains the transformed action for $(\tilde{\theta} - \tilde{t})$ twisting.

The above result can be generalized to WZNW models based on other groups G. If r is the rank of the group, and N denotes the total number of generators, then we can choose a parametrization of the group element (at least locally) of the form:

$$g = exp(i\sum_{i=1}^{r}\theta_{Li}H^i)exp(i\sum_{a=1}^{N-2r}\phi_a T^a)exp(i\sum_{j=1}^{r}\theta_{Rj}H^j) \tag{2.34}$$

where H^j are the generators of the Cartan subalgebra, and T^a denote a specific set of $N-2r$ generators outside the Cartan subalgebra. With this parametrization, using the Polyakov-Wiegman theorem (2.2), it can be seen that the background metric and the antisymmetric tensor field are independent of the $2r$ coordinates θ_{Li}, θ_{Ri} $(1 \le i \le r)$. Thus, there is an $O(2r, 2r)$ group of transformations which can generate new conformally invariant backgrounds. We expect that all the r^2 marginal deformations, generated by taking products of holomorphic and anti-holomorphic currents in the Cartan subalgebra, will be generated by these $O(2r, 2r)$ transformations.

The results of this and the previous section were extended by Henningson and Nappi [40] to general non-linear σ-models with abelian chiral isometries. They showed that if the theory has d_L chiral and d_R anti-chiral conserved abelian currents, then, all of its $d_L \times d_R$ marginal deformations of the type $\sum_{a,\bar{b}} C_{a\bar{b}} J^a \bar{J}^{\bar{b}}$ are generated by appropriately chosen $O(d_L+d_R, d_L+d_R)$ transformations. All the remaining $O(d_L+d_R, d_L+d_R)$ transformations have a trivial action on the theory.

2.4 O(3,3) TRANSFORMATIONS OF SU(2) ⊗ (SU(2)/U(1)) MODEL

The analysis so far gives us an understanding of the $O(d,d)$ transformations as current-current perturbations in conformally invariant non-linear σ-models with chiral isometries.

In the remaining part of this chapter we will generalize our study to coset models based on WZNW models where all the isometries are not chiral. $O(d,d)$ transformations involving these non-chiral isometries generate marginal deformations the existence of which cannot otherwise be inferred from the structure of the theory. The conformal field theory origin of the $O(d,d)$ deformations in these models is, therefore, not as straightforward as in models with chiral isometries. To put our analysis of this section in a proper perspective, we first formulate the problem in a general context. In [19], it was shown that any theory in D dimensions with d commuting isometries can be obtained from a theory in $D+d$ dimensions with d chiral and d anti-chiral commuting isometries by taking a coset with respect to d of the isometries. The non-trivial $O(d,d)$ transformations of the D-dimensional theory then correspond to gauging different linear combinations of the chiral currents in the $(D+d)$-dimensional theory. We refer to this as *twisted gauging*. However, if we consider a coset with respect to some $n < d$ of the chiral currents, then the resulting $(D+d-n)$-dimensional theory will have a larger class of deformations generated by the $O(2d-n, 2d-n)$ group, though a smaller number of them are now accounted for by the twisted gaugings of the original model. The residual current-current perturbations of the coset theory are also not sufficient to account for the extra deformations. The problem, therefore, is to understand the conformal field theory origin of the deformations of a coset model within the framework of the original ungauged theory [1]. We will address this problem in this section and section 6 by considering a specific example.

In this section, we will start with a conformal field theory that is a product of an $SU(2)$ WZNW model and an $SU(2)/U(1)$ coset model; and then construct the most general model obtained by $O(3,3)$ transformations of this model. It is clear that since the starting theory has only one pair of chiral currents, not all of its $O(3,3)$ deformations can be understood as $J\bar{J}$-type perturbations. To understand the origin of these deformations in relation to the parent theory, we discuss the gauging of deformed WZNW models in the next section. The $O(3,3)$ deformations of the coset are then related to the twisted gauging of the marginal deformations of the $SU(2) \otimes SU(2)$ theory in section 6.

[1]Obviously, if we do not restrict ourselves to remain within the framework of the original ungauged theory, then the deformations can again be explained by the mechanism of [19], by embedding the original theory itself in a theory with larger number of chiral currents.

The starting $SU(2)\otimes(SU(2)/U(1))$ model has the metric [51]

$$ds^2 = k_2(\frac{1}{4}dr^2 + \sin^2\frac{r}{2}dt^2 + \cos^2\frac{r}{2}d\tilde{t}^2) + k_1(\frac{1}{4}d\phi^2 + \tan^2\frac{\phi}{2}d\theta^2) \tag{2.35}$$

and the dilaton and the antisymmetric tensor fields

$$\Phi = -\ln\cos^2\frac{\phi}{2}, \qquad B_{t\tilde{t}} = -k_2\sin^2\frac{r}{2} \tag{2.36}$$

with all other components of the anti-symmetric tensor field being zero. The fields are independent of the three coordinates θ, t and $\tilde{t}$. As discussed in chapter 1, new conformally invariant backgrounds can be generated from the one given in equations (2.35), (2.36) via $O(3,3)$ transformations. A general $O(3,3)$ transformation can be written as a product of an $O(3)$ transformation and a three dimensional general coordinate transformation and gauge transformation involving the antisymmetric tensor field. We shall first perform the $O(3)$ transformation of the background given in eqs. (2.35), (2.36). This is generated by matrices of the form (1.12) with $\mathcal{R}=\mathcal{S}^{-1}$. In the present case ($d=3$), a general R may be taken to be of the form:

$$\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 \tag{2.37}$$

where,

$$\mathcal{R}_1 = \begin{pmatrix} \cos\tilde{\alpha} & -\sin\tilde{\alpha} & 0 \\ \sin\tilde{\alpha} & \cos\tilde{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{R}_2 = \begin{pmatrix} \cos\tilde{\beta} & 0 & \sin\tilde{\beta} \\ 0 & 1 & 0 \\ -\sin\tilde{\beta} & 0 & \cos\tilde{\beta} \end{pmatrix} \quad \mathcal{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tilde{\gamma} & -\sin\tilde{\gamma} \\ 0 & \sin\tilde{\gamma} & \cos\tilde{\gamma} \end{pmatrix} \tag{2.38}$$

Note that we are using the symbols $\tilde{\alpha}$, $\tilde{\beta}$ ... to denote the parameters of the $O(d,d)$ group that acts on the coset model, and the symbols α, β ... to denote the parameters of the $O(d,d)$ group that acts on the WZNW model. This notational distinction will be useful in the coming sections when we discuss the relationship between gauged $O(d,d)$ transformed WZNW models, and $O(d,d)$ transformed coset models. Calculation of the transformed fields is relatively straightforward. It turns out that the final metric obtained in this way is singular at $r=0$, $r=\pi$ and $\phi=0$ if we assume conventional periodicities in the variables θ, t and $\tilde{t}$. In order to remove these singularities, we need to perform a linear set of coordinate transformations of the form:

$$\begin{pmatrix} \theta \\ t \\ \tilde{t} \end{pmatrix} = V \begin{pmatrix} \theta' \\ t' \\ \tilde{t}' \end{pmatrix} \tag{2.39}$$

where V is a 3×3 matrix, such that the transformed metric satisfies the following conditions near $r=0$, $r=\pi$ and $\phi=0$:

$$\begin{aligned}
&\text{For } r\simeq 0: \qquad G'_{tt}\simeq k_2\frac{r^2}{4}\ ;\ G'_{t\tilde{t}}, G'_{t\theta}\propto r^2\\
&\text{For } r\simeq \pi: \qquad G'_{\tilde{t}\tilde{t}}\simeq k_2\frac{(r-\pi)^2}{4}\ ;\ G'_{\tilde{t}t}, G'_{\tilde{t}\theta}\propto (r-\pi)^2\\
&\text{For } \phi\simeq 0: \qquad G'_{\phi\phi}\simeq k_1\frac{\phi^2}{4}\ ;\ G'_{\phi t}, G'_{\phi\tilde{t}}\propto \phi^2
\end{aligned} \tag{2.40}$$

Such a metric is non-singular if we assume θ', t' and $\tilde{t}'$ to be angular coordinates with period 2π each. It turns out that these requirements fix the matrix V completely. The dilaton, metric and the antisymmetric tensor field components after the transformation are given by (for convenience of writing we have dropped the primes),

$$\begin{aligned}
&\Phi = -\ln\Delta\\
&G_{\theta\theta} = \Delta^{-1}k_1\sin^2\frac{\phi}{2}\{A^2\cos^2\frac{r}{2}+\sin^2\frac{r}{2}\}\\
&G_{tt} = \Delta^{-1}k_2\sin^2\frac{r}{2}\{B^2\sin^2\frac{\phi}{2}+A^2\cos^2\frac{\phi}{2}\}\\
&G_{\tilde{t}\tilde{t}} = \Delta^{-1}k_2\cos^2\frac{r}{2}\{C^2\sin^2\frac{\phi}{2}+\cos^2\frac{\phi}{2}\}\\
&G_{\theta t} = \Delta^{-1}\sqrt{k_2k_1}B\sin^2\frac{\phi}{2}\sin^2\frac{r}{2}\\
&G_{\theta\tilde{t}} = \Delta^{-1}\sqrt{k_2k_1}AC\sin^2\frac{\phi}{2}\cos^2\frac{r}{2}\\
&G_{t\tilde{t}} = 0\\
&B_{\theta t} = \Delta^{-1}\sqrt{k_2k_1}AC\sin^2\frac{\phi}{2}\sin^2\frac{r}{2}\\
&B_{\theta\tilde{t}} = \Delta^{-1}\sqrt{k_2k_1}B\sin^2\frac{\phi}{2}\cos^2\frac{r}{2}\\
&B_{t\tilde{t}} = -\Delta^{-1}k_2\sin^2\frac{r}{2}\{\cos^2\frac{\phi}{2}+C^2\sin^2\frac{\phi}{2}\}
\end{aligned} \tag{2.41}$$

where, A, B, C and Δ are given by

$$\begin{aligned}
A &= \frac{\cos\tilde\alpha\cos\tilde\beta\cos\tilde\gamma}{k_2\sin\tilde\gamma - \cos\tilde\alpha\cos\tilde\beta\cos\tilde\gamma} \\
B &= \sqrt{k_2 k_1}\frac{\sin\tilde\beta\cos\tilde\gamma}{k_2\sin\tilde\gamma - \cos\tilde\alpha\cos\tilde\beta\cos\tilde\gamma} \\
C &= \sqrt{k_2 k_1}\frac{\sin\tilde\alpha\cos\tilde\beta\cos\tilde\gamma}{k_2\sin\tilde\gamma - \cos\tilde\alpha\cos\tilde\beta\cos\tilde\gamma} \\
\Delta &= \cos^2\frac{\phi}{2}\{\sin^2\frac{r}{2} + A^2\cos^2\frac{r}{2}\} + \sin^2\frac{\phi}{2}\{B^2\cos^2\frac{r}{2} + C^2\sin^2\frac{r}{2}\}
\end{aligned} \tag{2.42}$$

Various special cases of this solution have been discussed in references [54, 62]. It is clear that only one combination of the three parameters A, B and C can be considered as arising from current-current perturbations of the ungauged $SU(2)$ in (2.35),(2.36). In section 6, we shall see that, for general A, B, C, these models represent conformal field theories obtained after gauging a marginally deformed $SU(2) \otimes SU(2)$ WZNW model. We start by discussing the gauging of deformations of the WZNW models of the type considered above.

2.5 GAUGING THE O(d, d) DEFORMED WZNW MODELS

In this section we discuss the gauging of commuting isometries in a non-linear σ-model where d of the coordinates are chosen to be along the isometry directions. When the isometries are related to chiral isometries, we describe a gauging which preserves the conformal invariance of the non-linear σ-model. We also give a prescription for obtaining this gauged model by a covariant derivative replacement. This formalism will be used in the next section to gauge the marginally deformed $SU(2)$ and $SU(2) \otimes SU(2)$ models.

Consider a non-linear σ-model with a Wess-Zumino term given by the action [2]

$$S = \frac{1}{2\pi}\int d^2z(\,g_{ij} + b_{ij}\,)\bar\partial X^i \partial X^j \tag{2.43}$$

[2] In order to make a distinction between the analysis of this section (dealing with gauging of σ-models) and the rest of this chapter (dealing with $O(d, d)$ transformations), we denote the fields G and B by g and b throughout this section.

To insure conformal invariance at the quantum level, the model may also contain a background dilaton field, Φ, which does not explicitly appear in the tree level action. Now, consider a set of vector fields $\xi_\alpha = \xi^i_\alpha(X)\partial_i$ which, on the σ-model manifold, generate motions of the type

$$\delta X^i = v^\alpha \xi^i_\alpha(X) \tag{2.44}$$

Here, v^α are global parameters. Under these transformations the variation of the action (2.43) is proportional to the Lie derivatives of the background fields along the vector fields ξ_α. The invariance of the action, then, leads to certain conditions which in the component form are written as

$$\xi^i_\alpha \partial_i \Phi = 0, \tag{2.45}$$

$$\nabla_i \xi_{j,\alpha} + \nabla_j \xi_{i,\alpha} = 0, \tag{2.46}$$

$$b_{ik}\,\partial_j \xi^k_\alpha + b_{kj}\,\partial_i \xi^k_\alpha + \partial_k b_{ij}\,\xi^k_\alpha = \partial_i K_{j,\alpha} - \partial_j K_{i,\alpha} \tag{2.47}$$

Equation (2.45) clearly means that the dilaton is constant along the integral curves corresponding to ξ_α. Equation (2.46) follows from the vanishing of the Lie derivative of g_{ij} and means that the metric admits Killing vectors ξ_α. The condition following from the invariance of the Wess-Zumino term is weaker and requires the existence of a 1-form $K_\alpha = K_{i,\alpha} dX^i$ the exterior derivative of which equals the Lie derivative of b_{ij} (2.47). If this condition is satisfied, the Wess-Zumino term is invariant upto a total derivative term. We refer to the transformations (2.44) satisfying the conditions (2.45)-(2.47) as the isometries of the non-linear σ-model. The associated isometry currents are given by

$$\begin{aligned} I_\alpha &= [\,(\,g_{ij} + b_{ij}\,)\xi^i_\alpha - K_{j,\alpha}\,]\partial X^j, \\ \bar{I}_\alpha &= [\,(\,g_{ij} - b_{ij}\,)\xi^i_\alpha + K_{j,\alpha}\,]\bar{\partial} X^j \end{aligned} \tag{2.48}$$

which, by virtue of the invariance of the action, are conserved: $\bar{\partial} I_\alpha + \partial \bar{I}_\alpha = 0$.

The gauging of the transformations (2.44) is not very straightforward due to the presence of the antisymmetric tensor field. In refs. [41, 42] it was shown that when the parameters v^α are local, then it is possible to construct a locally gauge invariant action by introducing a set of gauge fields provided the following conditions are satisfied

$$
\begin{aligned}
b_{jk}\,\xi^k_\beta\partial_i\xi^j_\alpha + \partial_j(b_{ik}\,\xi^k_\beta)\xi^j_\alpha + \partial_j K_{i,\beta}\xi^j_\alpha + K_{j,\beta}\partial_i\xi^j_\alpha &= f^\gamma_{\alpha\beta}(\,b_{ij}\xi^j_\gamma + K_{i,\gamma}), \\
K_{i,\alpha}\xi^i_\beta &= -K_{i,\beta}\xi^i_\alpha
\end{aligned}
\tag{2.49}
$$

Here, $f^\gamma_{\alpha\beta}$ are structure constants of the symmetry algebra defined by the Lie bracket of two vector fields, $[\,\xi_\alpha\,,\,\xi_\beta\,] = f^\gamma_{\alpha\beta}\,\xi_\gamma$. Since we will be interested only in commuting isometries, we will set $f^\gamma_{\alpha\beta} = 0$. After introducing gauge fields A^α_μ which transform as $A^\alpha_\mu \to A^\alpha_\mu + \partial_\mu v^\alpha$, the gauge invariant action is written as

$$
\begin{aligned}
S^{\text{gauged}} = \frac{1}{2\pi}\int d^2z[\,(\,g_{ij} \;\;+\;\; b_{ij}\,)\bar\partial X^i\partial X^j - A^\alpha\bar I_\alpha - \bar A^\alpha I_\alpha \\
+((\,g_{ij} \;\;+\;\; b_{ij}\,)\xi^i_\alpha\xi^j_\beta - \frac{1}{2}(K_{i,\alpha}\xi^i_\beta - K_{i,\beta}\xi^i_\alpha))\bar A^\alpha A^\beta\,]
\end{aligned}
\tag{2.50}
$$

This action has an arbitrariness stemming from the fact that equations (2.47) and (2.49) do not determine the 1-form $K_{i,\alpha}$ completely. Moreover, it cannot in general be obtained from (2.43) by a covariant derivative replacement. The arbitrariness of the gauged action turns out to be a virtue for the following reason. In situations of our interest the action in (2.43) is conformally invariant. Moreover, we want the coset theory, which is obtained from the gauged action (2.50) after gauge fixing and integrating out the gauge fields, also to be conformally invariant. This is in general not true. However, we show that if the action (2.43) has chiral isometries, we can fix the arbitrariness in the gauged action by choosing the 1-form K_α such that the gauge fields in (2.50) couple to the corresponding chiral currents. Then the resulting gauged theory may be described as a coset of the original conformal field theory by some $U(1)$ current algebra theories [18]. A generalization of the original argument of Goddard, Kent and Olive [43] still goes through and hence the resulting theory is again conformally invariant.

The $O(d,d)$ transformed WZNW models considered in the previous sections still have residual chiral invariances given by equations (2.21),(2.30). This enables us to break up the vector fields ξ_α generating the global isometries (2.44) into two sets, ξ_{La} and ξ_{Ra}, such that the isometries generated by them, *i.e.*,

$$
\delta_L X^i = v^a_L\xi^i_{La}\,, \qquad \delta_R X^i = v^a_R\xi^i_{Ra} \tag{2.51}
$$

are extendible to chiral isometries by choosing $v_L^a = v_L^a(z)$ and $v_R^a = v_R^a(\bar{z})$. The isometry currents I_{La}^μ and I_{Ra}^μ -where μ is a worldsheet index- associated with the above transformations are obtained from the expressions in (2.48) on replacing the pair of variables $(\xi_\alpha^i, K_{i,\alpha})$ by $(\xi_{La}^i, L_{i,a})$ and $(\xi_{Ra}^i, R_{i,a})$ respectively. If we choose to gauge a diagonal subgroup of (2.51) with $v_L^a = v_R^a = v^a(z,\bar{z})$, we can set $A_\mu^{La} = A_\mu^{Ra} \equiv A_\mu^a$. Then the gauged action is given by (2.50) with the index α replaced by the index a and, where, now

$$\xi_a^i = \xi_{La}^i + \xi_{Ra}^i \,, \qquad K_{i,a} = L_{i,a} + R_{i,a} \tag{2.52}$$

and, therefore, $I_a^\mu = I_{La}^\mu + I_{Ra}^\mu$. To fix the arbitrariness in this gauged action due to $L_{i,a}$ and $R_{i,a}$, we note that the chiral invariance of the ungauged action allows us to choose

$$L_{j,a} = -(\,g_{ij} - b_{ij}\,)\xi_{La}^i \qquad R_{j,a} = (\,g_{ij} + b_{ij}\,)\xi_{Ra}^i \tag{2.53}$$

satisfying equation (2.47). This gives $\bar{I}_{La} = I_{Ra} = 0$ and $\partial \bar{I}_{Ra} = \bar{\partial} I_{La} = 0$. We will denote these chiral currents by J_a, $\bar{J}_a$. If this choice is made in the gauged action, the gauge fields will couple to chiral currents of the ungauged theory, leading to a new conformal field theory as argued above. This can be checked explicitly in the specific examples we have, by verifying that the backgrounds obtained after gauge fixing and integrating out the gauge fields satisfy the β-function vanishing equations. Note that after obtaining $L_{j,a}$ and $R_{j,a}$ using (2.53), we need to verify that they satisfy the conditions (2.49). This further restricts the choice of ξ_{aL}^i and ξ_{aR}^i to anomaly free subgroups. In all the cases to be discussed, we will make appropriate choices of ξ_{aL}^i and ξ_{aR}^i so that these conditions are satisfied.

Next, we explore the possibility of writing the gauged action (2.50) in terms of covariant derivatives. This can be done, provided, by adding appropriate total derivative terms to the Lagrangian density, we can ensure that the Nöther currents associated with the chiral symmetries are the same as the chiral currents J_a, $\bar{J}_a$. If in the σ-model action (2.43) the field b_{ij} is replaced by $b_{ij} + 2\partial_{[i} w_{j]}$ and $\partial_\mu X^i$ by the covariant derivative $D_\mu X^i = \partial_\mu X^i - \xi_a^i A_\mu^a$, we get

$$S^{\text{gauged}} = \frac{1}{2\pi} \int d^2z (\,g_{ij} + b_{ij} + 2\partial_{[i} w_{j]}\,) \bar{D} X^i D X^j \tag{2.54}$$

This is the correct gauged action (2.50) provided one can find a w_j such that

$$2\partial_{[i} w_{j]}\, \xi_a^j = L_{i,a} + R_{i,a} \tag{2.55}$$

This is always possible if we restrict ourselves to backgrounds of the form (1.2) and to shift isometries in the coordinates on which the backgrounds do not depend ($\xi^\alpha_a = 0$, $\partial_i \xi^m_a = 0$). Equations (2.47) and (2.49) then imply that $L_{m,a}$ and $R_{m,a}$ are constant while $L_{\alpha,a} = R_{\alpha,a} = 0$. $2\partial_{[i} w_{j]}$ is, therefore, a constant matrix with non-zero elements only in the subspace spanned by the coordinates the isometries in which are gauged.

Combining (2.55) with equation (2.53) gives

$$2\partial_{[i} w_{j]}\, \xi^j_a = g_{ij}(\xi^j_{Ra} - \xi^j_{La}) - b_{ij}(\xi^j_{La} + \xi^j_{Ra}) \tag{2.56}$$

In the following we use equations (2.54) and (2.56) to gauge the shift isometries of $O(d,d)$ transformed WZNW models.

2.6 APPLICATION TO MARGINALLY DEFORMED WZNW MODELS

Using the gauging procedure for σ-models with chiral isometries, we are in a position to further investigate the properties of the $O(d,d)$ transformations. In this section, we will first consider the gauging of the $O(2,2)$ transformed $SU(2)$ WZNW model. The result will be used to give an explanation, in the context of the $SU(2)$ theory, for the observation made in [61]. In this reference, it was shown that for theories with chiral currents [18, 19], the $U(1)$ cosets plus some free bosons, can be obtained by certain fixed $O(d,d)$ transformations. Next, we consider the gauging of the deformed $SU(2) \otimes SU(2)$ theory. This result will be used to understand, as promised in section 5, the conformal field theory origins of the deformations of the $SU(2) \otimes (SU(2)/U(1))$ theory.

Let us consider the gauging of the $O(2,2)$ deformed $SU(2)_k$ model given in (2.17). For this model there are two gaugings possible: i) the axial gauging, in which the translations of the coordinate $\widetilde{\theta}$ are made local and ii) the vector gauging, in which the translations of the coordinate θ are made local. We first consider the axial gauging. Reading out the Killing vectors ξ^i_L and ξ^i_R from (2.21) and substituting in (2.56), we get $\partial_{[\theta} w_{\tilde{\theta}]} = 0$, This is a consequence of the judicious choice of the constant L in (2.16). The gauged action can now be directly obtained by a covariant derivative replacement and is given by

$$S^{\rm gauged}_{(\alpha)}(\phi,\theta,\widetilde{\theta},A) = S_{(\alpha)}(\phi,\theta,\widetilde{\theta}) - \frac{1}{2\pi}\int d^2z\,(\,A\bar{J}_{(\alpha)} + \bar{A}J_{(\alpha)} - \frac{k\cos^2\frac{\phi}{2}}{\Delta\cos^2\alpha}A\bar{A}\,) \tag{2.57}$$

where $J_{(\alpha)}$, $\bar{J}_{(\alpha)}$ have been defined in (2.22). However, on fixing the gauge ($\tilde{\theta} = 0$) and integrating out the gauge field, one finds that the deformation parameter α disappears from the coset model. One, thus, ends up with the well-known $SU(2)/U(1)$ axial coset given by [51, 53]

$$ds^2 = (k/4)d\phi^2 + k\tan^2(\phi/2)\,d\theta^2\,, \qquad e^{-\Phi} = \cos^2\frac{\phi}{2} \tag{2.58}$$

The reason for the disappearance of α lies in the freedom to redefine the gauge fields. In fact, substituting

$$A = \cos\alpha\sqrt{\Delta}A' + \cos\alpha(\cos\alpha - \sqrt{\Delta})\frac{J}{k\cos^2\frac{\phi}{2}} \tag{2.59}$$

in (2.57), with a similar expression for $\bar{A}$, we find that the gauged action is independent of α :

$$S^{\text{gauged}}_{(\alpha)}(\phi,\theta,\tilde{\theta},A) = S^{\text{gauged}}_{(\alpha=0)}(\phi,\theta,\tilde{\theta},A') \tag{2.60}$$

The Jacobian of the transformation modifies the path integral measure and readjusts the dilaton field to zero.

This result may also be understood in a qualitative manner as follows. Note that gauging the deformed WZNW model corresponds to taking a coset of the ungauged conformal field theory by a $U(1)$ current algebra theory. In the coset theory, the action of the operators $J_{(\alpha)}$, $\bar{J}_{(\alpha)}$ becomes trivial. Hence, one would expect that the effect of perturbing the original theory by the $J_{(\alpha)}\bar{J}_{(\alpha)}$ operator before gauging will be washed out after gauging. This, in turn, implies that the final theory will be independent of the parameter α, which is indeed the case here.

The analysis of the vector gauging of the $O(2,2)$ deformed $SU(2)$ model is similar to the axial case. The only difference is that, in this case, in order to obtain the gauged action by a covariant derivative substitution, the constant L given in equation (2.16) has to be replaced by

$$L = \cos\alpha(k\cos\alpha + \sin\alpha) - k \tag{2.61}$$

This dose not alter the analysis of section 2. In this case, after gauge fixing ($\theta = 0$) and integrating out the gauge field, one obtains the vector $SU(2)/U(1)$ coset

$$ds^2 = (k/4)d\phi^2 + k\cot^2(\phi/2)\,d\tilde{\theta}^2\,, \qquad e^{-\Phi} = \sin^2\frac{\phi}{2} \tag{2.62}$$

It is easy to check that for $\alpha = -\pi/2$, the deformed action given by equation (2.17) reduces to the axial coset (2.58) plus a compact free boson of zero radius. Similarly, for $\tan\alpha = 1/k$, the deformed theory reduces to the vector coset (2.62) plus a free boson of zero radius. In fact one can easily see that it is sufficient to consider α in the range $-\pi/2 \leq \alpha \leq \tan^{-1}(1/k)$. Any value of the parameter outside this range is equivalent to some value inside the range. Also, a study of the marginal deformations, as given in equation (2.23), shows that near the endpoints of the range of α, *i.e.*, $\alpha = -\pi/2$ and $\alpha = \tan^{-1}(1/k)$, the "speed" of marginal deformations becomes progressively smaller until it reaches zero at the end points which correspond to the axial and vector coset models. Thus, the marginal deformation line of the original $SU(2)_k$ WZNW model contains the axial and vector $SU(2)/U(1)$ cosets (plus free bosons) as limit points. This phenomenon has been interpreted as a continuous topology change in the context of the $SL(2,R)$ theory[44].

These results are consistent with the observation of [61] that, in non-linear σ-models with chirally conserved currents [18, 19], certain fixed $O(d,d)$ transformations produce a coset of the original theory plus free bosons. We are now in a position to explain the origin of this observation [26]. The question to answer is : why should an $O(d,d)$ transformation give rise to a coset of the original theory? We try to understand this in the context of our deformed $SU(2)$ theory. To understand why the coset models are obtained by fixed $O(2,2)$ transformations, we use equation (2.60) which states that the gauged model is independent of the parameter α. We can, therefore, start from a theory with any value of this parameter before introducing the gauge fields. In particular, we can look for some $\alpha = \alpha_0$ for which one of the coordinates decouples as a free field. For example, in our case we can choose $\alpha = -\pi/2$ for which $\tilde{\theta}$, after appropriate scaling, decouples as a free field:

$$S_{(\alpha=\alpha_0)}(\phi,\theta,\tilde{\theta}) = S'_{(\alpha=\alpha_0)}(\phi,\theta) + \frac{1}{2\pi}\int d^2z\bar{\partial}\tilde{\theta}\partial\tilde{\theta} \tag{2.63}$$

Now, we consider the axial $U(1)$ gauging of the above model. On the right hand side of the above equation, only the term containing $\tilde{\theta}$ gets affected by gauging. Gauge fixing and integrating out the gauge field simply eliminates this term leaving $S'_{(\alpha=\alpha_0)}(\phi,\theta)$ unchanged. On the other hand, by equation (2.60), the left hand side in (2.63) after gauging must be the same as $S^{\rm gauged}_{(\alpha=0)}(\phi,\theta,\tilde{\theta},A')$ which leads to the axial coset model. This argument shows that once we know that $S_{\alpha=\alpha_0}$ can be written in the form of equation (2.63), then

$S'_{\alpha=\alpha_0}(\phi,\theta)$ must be the action of the gauged WZNW model (in our case, the $SU(2)/U(1)$ coset model). The vector gauging can also be understood along the same lines. Though the situation considered by Kumar [61] is more general, the results can be understood in the same way. Note that the above construction requires that the $O(d,d)$ transformed theory is written in a form that can be gauged by covariant derivative replacement. A situation related to [61] was considered in [62] where it was observed that by adding a free field to the singular $SL(2,R)/U(1)$ background and an $O(2,2)$ transformation, one can "boost away" the singularity. This can be understood by inverting the argument based on equation (2.63): the singularities which are boosted away are the ones that appear as a consequence of a $U(1)$ gauging of the boosted theory, for which (2.60) holds. The argument can also be generalized to the case of the 4-dimensional solution of reference [63] also considered in [62].

Next, we consider the gauging of the $O(4,4)$ transformed $SU(2)_{k_1}\otimes SU(2)_{k_2}$ model. In studying this model, our aim is to answer the problem posed in section 5 regarding the origin of the $O(3,3)$ deformations of the $SU(2)\otimes(SU(2)/U(1))$ theory. We will show that a parameter in the deformed coset model which does not have its origin in the current-current perturbations of the coset or in twisted gaugings of the parent $SU(2)\otimes SU(2)$ theory, can be identified as following from a marginal deformation of the parent theory. Following the strategy of section 3, we consider the gauging of deformations of the $SU(2)_{k_1}\otimes SU(2)_{k_2}$ model generated by different subgroups of the the $O(4,4)$ group. First, we consider the case in which the two $SU(2)$ sectors are not mixed by $O(4,4)$. This corresponds to the action given in (2.25). As shown above, a naive gauging of a $U(1)$ subgroup belonging to either of the two $SU(2)$'s will simply give the direct product of a marginally deformed $SU(2)$ WZNW model and an $SU(2)/U(1)$ coset model. This is clearly contained in the $O(3,3)$ deformations of the backgrounds given in (2.35),(2.36). To obtain a non-trivial result, we gauge a $U(1)$ subgroup which acts simultaneously on both sectors of the theory. The action of this subgroup is given by

$$\delta\theta=0\,,\quad \delta\tilde{\theta}=\frac{v(z,\bar{z})}{\cos^2\alpha}\,,\quad \delta t=0\,,\quad \delta\tilde{t}=\lambda\frac{v(z,\bar{z})}{\cos^2\beta} \tag{2.64}$$

where, λ is a new parameter. Notice that λ is the parameter which gives rise to the twisted gauging of the theory. It therefore corresponds to one of the $O(3,3)$ deformation parameters of the coset theory as mentioned in section 4 [19]. The gauged action invariant under the above transformation is now given by

$$S^{\rm gauged}_{(\alpha,\beta)} = S_{(\alpha)}(\,\phi,\theta,\widetilde{\theta}\,) \;\; + \;\; S_{(\beta)}(\,r,t,\widetilde{t}\,) - \frac{1}{2\pi}\int d^2z\Big[\,A\,(\bar{J}_{1\,(\alpha)} + \lambda\,\bar{J}_{2\,(\beta)})$$

$$+\bar{A}\,(J_{1\,(\alpha)} \;\; + \;\; \lambda\,J_{2\,(\beta)}) - A\bar{A}\,(\,\frac{\cos^2\frac{\phi}{2}}{\cos^2\alpha\Delta_1} + \lambda^2\frac{\cos^2\frac{r}{2}}{\cos^2\beta\Delta_2}\,)\Big] \qquad (2.65)$$

where, the subscripts 1 and 2 refer to the theories $(\,\phi,\theta,\widetilde{\theta}\,)$ and $(\,r,t,\widetilde{t}\,)$, respectively. The coset model is obtained after fixing the gauge $\widetilde{\theta} = 0$ and integrating out the gauge fields. The new background fields are given by

$$\begin{aligned}
G_{\theta\theta} &= D^{-1}k_1\sin^2\frac{\phi}{2}(\sin^2\frac{r}{2} + Q\cos^2\frac{r}{2}) \\
G_{tt} &= D^{-1}k_2\sin^2\frac{r}{2}(k_1k_2P^2\sin^2\frac{\phi}{2} + Q\cos^2\frac{\phi}{2}) \\
G_{\tilde{t}\tilde{t}} &= D^{-1}k_2\cos^2\frac{\phi}{2}\cos^2\frac{r}{2} \\
G_{\theta t} &= D^{-1}k_1k_2P\sin^2\frac{\phi}{2}\sin^2\frac{r}{2} \\
G_{\theta\tilde{t}} &= G_{t\tilde{t}} = B_{\theta t} = 0 \\
B_{\theta\tilde{t}} &= D^{-1}k_1k_2P\sin^2\frac{\phi}{2}\cos^2\frac{r}{2} \\
B_{t\tilde{t}} &= -D^{-1}k_2\sin^2\frac{r}{2}\cos^2\frac{\phi}{2} \\
\Phi &= -\ln D
\end{aligned} \qquad (2.66)$$

Here, the parameters P and Q and the field dependent quantity D are defined as

$$\begin{aligned}
P &= \frac{\lambda}{k_1}\left(\frac{\cos\alpha}{\cos\beta}\right)^2 \\
Q &= (1 - k_2\tan\beta)^2 + \lambda^2\frac{k_2}{k_1}\left(\frac{\cos\alpha}{\cos\beta}\right)^4(1 - k_1\tan\alpha)^2 \\
D &= \cos^2\frac{\phi}{2} + k_1k_2P^2\cos^2\frac{r}{2} - (1 - Q + k_1k_2P^2)\cos^2\frac{\phi}{2}\cos^2\frac{r}{2}
\end{aligned} \qquad (2.67)$$

Note that the background fields depend on only two independent parameters P and Q. The reason for the disappearance of the third parameter is again to be sought in the freedom to redefine the gauge fields. In fact, if A and $\bar{A}$ are redefined such that the terms linear in the gauge fields in (2.65) are eliminated, the resulting action will depend only on P and Q. Now we can compare the backgrounds in (2.41) and (2.66) . In fact, if we put $C = 0$ in (2.41), and identify P and Q of (2.66) as

$$P = \frac{B}{\sqrt{k_2 k_1}}\,, \quad Q = A^2 \tag{2.68}$$

then the two sets of background fields in (2.41) and (2.66) turn out to be the same.

The background fields obtained on gauging the $\theta - t$ twisted model are still of the form (2.66). Thus, in order to generate a class of solutions (2.41) with non-zero C, we consider the gauging of the marginally deformed $SU(2) \otimes SU(2)$ theory obtained by an $O(2,2) \subset O(4,4)$ which twists θ and $\tilde{t}$. The action $S'_{(\gamma)}$, chiral symmetries, and chiral currents, $J'_{i\,(\gamma)}$ and $\bar{J}'_{i\,(\gamma)}$, of this theory are obtained from equations (2.27),(2.30) and (2.31) by the replacements $r \to r+\pi$ and $t \leftrightarrow \tilde{t}$. The transformation to be gauged is $\delta\theta = 0$, $\delta\tilde{\theta} = v(z,\bar{z})$, $\delta t = \lambda v(z,\bar{z})$ and $\delta\tilde{t} = 0$. From (2.56) we get $\partial_{[\theta} w_{\tilde{\theta}]} = -k_1$, $\partial_{[t} w_{\tilde{t}]} = k_2$ which, using (2.54), leads to the following gauged action

$$\begin{aligned} S'^{\rm gauged}_{(\gamma)} = S'_{(\gamma)} - \frac{1}{2\pi}\int d^2z \Big[& A\,(\bar{J}'_{1\,(\gamma)} + \lambda\,\bar{J}'_{2\,(\gamma)}) + \bar{A}\,(J'_{1\,(\gamma)} + \lambda\,J'_{2\,(\gamma)}) \\ & - \frac{1}{\Delta} A\bar{A}\,\Big(k_1 \cos^2\frac{\phi}{2}\,(\cos^2\gamma + k_1 k_2 \sin^2\gamma\, \cos^2\frac{r}{2}) \\ & + \lambda^2\, k_2 \sin^2\frac{r}{2}\,(\cos^2\gamma + k_1 k_2 \sin^2\gamma\, \sin^2\frac{\phi}{2})\Big)\Big] \end{aligned} \tag{2.69}$$

where,

$$\Delta = \cos^2\gamma + k_1 k_2 \sin^2\gamma \sin^2\frac{\phi}{2}\cos^2\frac{r}{2} \tag{2.70}$$

On fixing the gauge $\tilde{\theta} = 0$ and integrating out the gauge fields, we obtain the following expressions for the metric, antisymmetric tensor field and the dilaton field :

$$\begin{aligned} G_{\theta\theta} &= D^{-1} k_1 \sin^2\frac{\phi}{2}[(1 + k_1 k_2 M N^2)\sin^2\frac{r}{2} + M\cos^2\frac{r}{2}) \\ G_{tt} &= D^{-1} k_2 M \sin^2\frac{r}{2}\cos^2\frac{\phi}{2} \\ G_{\tilde{t}\tilde{t}} &= D^{-1} k_2 \cos^2\frac{r}{2}[k_1 k_2 M N^2 \sin^2\frac{\phi}{2} + (1 + k_1 k_2 M N^2)\cos^2\frac{\phi}{2}] \\ G_{\theta\tilde{t}} &= D^{-1} k_1 k_2 M N \sin^2\frac{\phi}{2}\cos^2\frac{r}{2} \\ G_{\theta t} &= G_{t\tilde{t}} = B_{\theta\tilde{t}} = 0 \end{aligned}$$

$$B_{\theta t} = D^{-1} k_1 k_2 M N \sin^2 \frac{\phi}{2} \sin^2 \frac{r}{2}$$

$$B_{t\tilde{t}} = D^{-1} k_2 M \cos^2 \frac{\phi}{2} \cos^2 \frac{r}{2}$$

$$\Phi = -\ln D \tag{2.71}$$

where,

$$D = k_1 k_2 M N^2 + \cos^2 \frac{\phi}{2} - k_1 k_2 M N^2 \cos^2 \frac{r}{2} + (M-1) \cos^2 \frac{\phi}{2} \cos^2 \frac{r}{2}$$

$$M = \frac{\cos^2 \gamma + k_1 k_2 \sin^2 \gamma}{\cos^2 \gamma - \lambda^2 k_2 \sin^2 \gamma} \tag{2.72}$$

$$N = \lambda / k_1$$

These fields can also be obtained from (2.66) after replacing r by $r + \pi$ and interchanging t and $\tilde{t}$. This corresponds to gauging a combination of axial and vector $U(1)$'s. To compare the two sets of background fields in (2.41) and (2.71) (up to an overall shift of the dilaton field), we have to set $B = 0$ in (2.42) and $B_{t\tilde{t}} \to B_{t\tilde{t}} + k_2$ in (2.41) and identify its remaining parameters in terms of the parameters M and N of (2.71) as

$$M = \frac{A^2}{1 - C^2}, \quad N = \frac{C}{A\sqrt{k_2 k_1}} \tag{2.73}$$

This shows that various classes of solutions of the low energy equations of motion which are obtained by the action of the $O(3,3)$ group on the $(SU(2)/U(1)) \otimes SU(2)$ coset model, can be constructed as $U(1)$ cosets of the marginally deformed $SU(2) \otimes SU(2)$ WZNW theory.

2.7 SUMMARY

In this chapter, we have studied σ-models which are obtained by $O(d,d)$ transformations of the $SU(2)_k$ and $SU(2)_{k_1} \otimes SU(2)_{k_2}$ WZNW models and have shown that they correspond to finite marginal deformations of the original WZNW theories. These marginal deformations are generated by a product of holomorphic and anti-holomorphic currents belonging to the Cartan subalgebra of the underlying current algebra. We have also studied the gauging of $U(1)$ subgroups of these marginally deformed theories. For the $SU(2)_k$ theory, we have shown that its axial and vector $U(1)$ cosets (plus a compact free boson) are limit points on the deformation line of the $SU(2)_k$ theory. For the $(SU(2)/U(1)) \otimes SU(2)$ model, we have shown that its deformations which are not accounted for by the known marginal operators in the theory are related to the marginal deformations of the parent $SU(2) \otimes SU(2)$ theory.

Our analysis enables us to give a σ-model description of finite marginal deformations of WZNW models and the cosets based on these deformed theories. The existence of marginally perturbed models was proved to all orders in perturbation parameter in [39], where, it was shown that marginal perturbations by products of holomorphic and anti-holomorphic currents from the Cartan subalgebra are integrable. However, though, it is easy to write down the form of these models to first order in the perturbation parameter, obtaining the σ-model for a finite deformation is in no way a straightforward task without the help of the $O(d,d)$ transformations.

Although the existence of the σ-model actions describing the deformed WZNW models and their gauging is exact to all orders in the deformation parameter, the explicit expressions that we have obtained are valid only to the lowest order in the derivatives (in the target space). In other words, the σ-model β-functions are zero only to one loop order, or in the large k limit. This limitation stems from the fact that the explicit $O(d,d)$ transformation laws of various fields are known only to this order, though, corrections have been computed in some cases [27, 28, 29]. The existence of $O(d,d)$ transformations that converts a conformally invariant background to another conformally invariant background in the general situation has, however, been established to all orders [16, 24, 19] in the σ-model perturbation theory. This, in turn, shows the existence of σ-models which represent deformed WZNW models (and their cosets) to all orders in the σ-model loop expansion.

Chapter 3

THE O(d, d + p) TRANSFORMATIONS IN HETEROTIC STRING THEORY

3.1 INTRODUCTION

In this chapter we will show that in heterotic string theory the $O(d) \otimes O(d)$ group of transformations introduced in Chapter 1, in the context of bosonic and superstring theories, can be enlarged to an $O(d-1,1) \otimes O(d+p-1,1)$ group. A novel feature of this enlarged group is that, acting on the background fields of heterotic string theory, it can relate configurations with zero gauge field to the ones with non-trivial gauge fields. An explicit example of this will also be discussed. The content of this chapter is based on the work appearing in reference [23]. We begin by discussing some relevant features of heterotic string theory.

The heterotic string theory is a realistic candidate for a unified theory of fundamental interactions. The consistency of this theory depends crucially on the fact that, in a string theory, the conformal anomalies in the left-moving and the right-moving sectors are independent of each other and, therefore, can be made to vanish separately. This allows us to construct the heterotic string as a hybrid of the 26-dimensional bosonic string in the right-moving sector and the 10-dimensional superstring in the left-moving sector. This theory has $(1,0)$ supersymmetry on the worldsheet and $N = 1$ supersymmetry in the target space. The first 10 of the bosonic string coordinates are combined with the superstring coordinates, giving rise to the space-time coordinates of the heterotic string. The remaining 16 right-moving bosonic coordinates are compactified on a 16 dimensional torus generated by a 16 dimensional lattice, with the associated momenta lying on the dual lattice. One-loop modular invariance of the partition function requires the momentum lattice to be Euclidean, even and self-dual. Such a lattice corresponds to either the root lattice of the group $E_8 \otimes E_8$ or the weight lattice of $Spin(32)/\mathbf{Z}_2$ which contains the root lattice of $SO(32)$. The massless spectrum of the theory consists of states in the gravity sector and states in the gauge sector. The gravity sector states are analogous to the massless states in superstring theory and consist of the graviton, the antisymmetric tensor field and the dilaton along with their fermionic super-partners, the gravitino and the dilatino. The gauge sector has two types of states: First, there are 16 Kaluza-Klein type vector bosons, one for each compactified coordinate. The vertex operators for these states are constructed entirely in terms of the oscillators and do not depend on the lattice momenta. However, one can also construct massless states with non-zero lattice momentum and winding. In fact, considering the lattice as associated with a simply laced group G, we can construct one massless boson corresponding to each root vector of G [74, 75]. Thus, for both $G = E_8 \otimes E_8$ and $SO(32)$, the 480 vector bosons constructed this way, along with the 16 Kaluza-Klein modes, fill up the adjoint representation of the group G. The massless spectrum also contains 496 gauginos. Let us choose a Cartan basis for the Lie algebra of G such that the Kaluza-Klein modes are $U(1)^{16}$ gauge fields belonging to the Cartan subalgebra. Then, the structure constants appearing in the commutators between elements from the Cartan subalgebra and the rest of the algebra are the components of the root vectors which are now nothing but the components of the lattice momentum vectors along the 16 compactified directions. Thus, the 16 dimensional lattice momentum corresponding to a given state can be interpreted as the 16 electric charges associated with the 16 $U(1)$ gauge fields.

As before, one can consider the propagation of the heterotic string in a background of the above massless excitations. The background fields now consist of the graviton G_{MN}, the antisymmetric tensor field B_{MN}, the dilaton Φ and the gauge field $A^a_M(X)$, where, a is the gauge group index. For definiteness we assume the gauge group to be $E_8 \otimes E_8$ though all our results apply equally well to $SO(32)$ without any modification. We again set the fermionic backgrounds to zero. The presence of the gauge field modifies the low-energy equations of motion and, therefore, the low energy effective action in more than one way. The obvious modification is the addition of a gauge field kinetic term to the effective action.

A more subtle modification arises from the left-right asymmetry of the theory and results in a change in the definition of the 3-form H_{MNP} by the addition of a gauge Chern-Simons term. The low-energy effective action for the heterotic string theory thus becomes

$$S = - \int d^D x \sqrt{-\det G} e^{-\Phi} \times \left(\Lambda - R^{(D)}(G) + \frac{1}{12} H_{MNP} H^{MNP} - G^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{8} \sum_a F^a_{MN} F^{aMN} \right) \quad (3.1)$$

where $R^{(D)}$ denotes the D dimensional Ricci scalar, and Λ is the cosmological constant equal to $(D-10)/2$ for heterotic string. The gauge field strength is given, as usual, by $F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + f^{abc} A^a_M A^b_N$ and the 3-form H_{MNP} is defined as

$$H_{MNP} = \partial_M B_{NP} + \text{cyclic permutations} + (\Omega^{(3)}_L)_{MNP} - (\Omega^{(3)}_A)_{MNP} \quad (3.2)$$

Here, $\Omega^{(3)}_A$ is the gauge Chern-Simons term given by $(\Omega^{(3)}_A)_{MNP} = (1/4)(A^a_M F^a_{NP} +$ cyclic permutations $- f^{abc} A^a_M A^b_N A^c_P)$ and $\Omega^{(3)}_L$ is the Lorentz Chern-Simons term defined similar to the gauge Chern-Simons term with the gauge field connection A_M replaced by the spin connection ω_M. The Lorentz Chern-Simons term is, however, a higher derivative term compared to its gauge counterpart and is not relevant to our analysis here which is accurate to one-loop in the σ-model perturbation theory.

As discussed in Chapter 1, it has been shown previously [11]-[17] that in any string theory, if we look for solutions that are independent of d of the space-time coordinates $\hat{X}^m$, then the space of such solutions has an $O(d) \otimes O(d)$ or $O(d-1,1) \otimes O(d-1,1)$ symmetry, depending on whether the coordinates $\hat{X}^m$ have Euclidean or Minkowski signature. (For definiteness, we shall assume from now on that the coordinates $\hat{X}^m$ have Minkowski signature). To these transformations we can add general linear transformations in d dimensions and gauge transformations of the antisymmetric tensor field, to elevate the symmetry group to an $O(d,d)$ group. Under the action of this $O(d-1,1) \otimes O(d-1,1)$ transformation a given solution is in general mapped to an inequivalent solution. In this chapter we will generalize these results to the heterotic string theory. We will show that in heterotic string theory, if we look for solutions that are independent of d of the space-time coordinates, and for which the background gauge field lies in a subgroup that commutes with p of the $U(1)$ generators of the gauge group, then the space of such solutions has an $O(d-1,1) \otimes O(d+p-1,1)$

symmetry [1]. These transformations, when augmented with d-dimensional general coordinate transformations and gauge transformations of the antisymmetric tensor field and the gauge field, give rise to an $O(d, d+p)$ group of transformations. We shall also use the $O(d-1,1)\otimes O(d+p-1,1)$ transformations to generate new classical solutions of heterotic string theory starting from the known ones.

The organization of this chapter is as follows : In section 2 we present general arguments to show the existence of an $O(d-1,1)\otimes O(d+p-1,1)$ symmetry in the heterotic string theory for the restricted class of backgrounds described above. In section 3 we study the manifestation of this symmetry in the low- energy effective field theory and give the explicit transformations of the background fields. Section 4 contains an application of this $O(d-1,1)\otimes O(d+p-1,1)$ transformation to a known solution of the heterotic string theory, namely, the black p-brane solution in ten dimensions carrying magnetic charge [77]. We show that by applying the $O(d-1,1)\otimes O(d+p-1,1)$ transformation on this solution we can generate a black p-brane solution carrying electric, magnetic and antisymmetric tensor field (axionic) charges. We summarize our results in section 5.

3.2 EXISTENCE OF $\mathrm{O(d-1,1)\otimes O(d+p-1,1)}$ SYMMETRY IN HETEROTIC STRING THEORY

In this section we will discuss the origin of the $O(d-1,1)\otimes O(d+p-1,1)$ symmetry in heterotic string theory with properly restricted background fields. We will first describe the factorization properties of a restricted class of correlation functions and then use this property to argue the existence of an $O(d-1,1)\otimes O(d+p-1,1)$ symmetry of the space of classical vacua. To focus our ideas, we perform the analysis in the framework of string field theory though the arguments remain valid in the first quantized theory as well.

As mentioned in Chapter 1, the origin of the $O(d-1,1)\otimes O(d-1,1)$ transformations can be traced to the fact [16] that if we restrict ourselves to backgrounds that are independent of d coordinates $\hat{X}^m$, then the interactions involving such backgrounds are described by correlation functions of vertex operators which carry zero momenta conjugate to the

[1] Similar symmetries had been seen earlier in the context of supergravity theories [70], and invariance of the classical equations of motion in the two dimensional σ-model under such transformations of the background was shown in [71, 72] following an earlier work of Gaillard and Zumino [73].

coordinates $\hat{X}^m$ in the two dimensional non-linear σ-model. A basis of such vertex operators can be chosen in such a way that each element of the basis involves a product of three terms, one term is only a function of the holomorphic derivatives of $\hat{X}^m$, another a function of anti-holomorphic derivatives of $\hat{X}^m$ and the third one is independent of $\hat{X}^m$ but has a dependence on the other fields. The correlation functions of such vertex operators also factorizes into products of three terms, the correlation function of the $\hat{X}^m$ independent parts the the correlation functions in the holomorphic sector and the antiholomorphic sector. Each of the last two correlation functions is separately invariant under the Lorentz transformations involving the coordinates $\hat{X}^m$. In other words, these correlation functions have an $O(d-1,1) \otimes O(d-1,1)$ symmetry. As a result, the action involving $\hat{X}^m$ independent background fields also has an $O(d-1,1) \otimes O(d-1,1)$ symmetry. In heterotic string theory, besides the usual space-time coordinates we also have 16 internal coordinates which have only right moving (anti-holomorphic) components but no left moving (holomorphic) components. If we consider backgrounds which are independent of d of the space-time coordinates, and carry zero momentum (and winding number) in p of the 16 internal coordinates, then the previous argument can be generalized easily to conclude that the space of such solutions should have an $O(d-1,1) \otimes O(d+p-1,1)$ symmetry. For the sake of clarity, we shall now present this argument in some detail by starting with a description of the factorization properties of the correlation functions associated with the above restricted backgrounds.

The argument is best presented in the language of string field theory. Let us first assume that there is an underlying string field theory that governs the dynamics of heterotic string, and that the vertices in this string field theory are given in terms of correlation functions in the conformal field theory describing the first quantized heterotic string theory. We consider heterotic string theory formulated in 10 dimensional flat space-time, although the argument can easily be generalized to the case where some of the dimensions are replaced by an arbitrary (1,0) super-conformal field theory of the correct central charge. Let $\{\Phi_a\}$ denote the set of basis states in the conformal field theory, including the ghost sector. The string field is given by $|\Psi\rangle = \sum_a \psi_a |\Phi_a\rangle$, with ψ_a's as the dynamical variables of the string field theory. Then the general N-point interaction vertex is given by $\psi_{a_1} \ldots \psi_{a_N} f_N(a_1, \ldots a_N)$, where $f_N(a_1, \ldots a_N)$ denotes a quantity constructed out of an N- point correlation function of some conformal transform of the fields $\Phi_{a_1}, \ldots \Phi_{a_N}$.[2] Now consider the string field configurations which are independent of d of the space-time coordinates $\hat{X}^m$ and which carry zero momentum in p (out of 16) of the internal coordinates $\check{X}^R$, with $1 \leq R \leq p$. For such configurations, only those components ψ_a will be non-zero for which the corresponding basis states $|\Phi_a\rangle$ have zero momentum along the d space-time directions, and also zero momentum along the p internal directions. A basis of such states in the conformal field theory can be chosen in the form $|\chi_l\rangle \otimes |\bar{\chi}_{\bar{l}}\rangle \otimes |\Phi'_{a'}\rangle$, where $|\chi_l\rangle$ and $\bar{\chi}_{\bar{l}}\rangle$ are the basis states in the holomorphic and the antiholomorphic sectors respectively of the conformal field theory described by the coordinates $\hat{X}^m$ and $\check{X}^R$, and $|\Phi'_{a'}\rangle$ denote the basis of states in the conformal field theory describing the rest of the system. The reason for this decomposition is that, in a generic vertex operator, the factor which is sensitive to the string center of mass coordinates and, therefore, is not holomorphiclly decomposable, is of the form $e^{k.X}$. However, if some component of the momentum, say k_i, is zero, then this factor does not spoil the factorizability of the X^i dependent part of the vertex operator. The correlation functions involving the above basis states on the sphere factorize into correlation functions involving the states $|\chi_l\rangle$, correlation functions involving the states $|\bar{\chi}_{\bar{l}}\rangle$ and correlation functions involving the states $|\Phi'_{a'}\rangle$. Thus f_N also has the factorized form:

$$f_N(l_1, \bar{l}_1, a'_1, \ldots l_N, \bar{l}_N, a'_N) = f_N^{(1)}(l_1, \ldots l_N) f_N^{(2)}(\bar{l}_1, \ldots \bar{l}_N) f_N^{(3)}(a'_1, \ldots a'_N) \qquad (3.3)$$

We now note that $f^{(1)}$ and $f^{(2)}$ are separately invariant under $O(d-1,1)$ and $O(d-1+p,1)$ transformations, respectively, which simply correspond to Lorentz transformations acting on the coordinates $\hat{X}^m$ and $\check{X}^R$. Note that though the general correlation functions in the conformal field theory describing the internal coordinates $\check{X}^R$ have no rotational symmetry due to the fact that the compactification torus is not invariant under an arbitrary rotation, in the zero momentum sector the correlation functions are completely ignorant of the compactification of the coordinates $\check{X}^R$. As a result of this, the correlation functions are not only symmetric under a rotation among the coordinates $\check{X}^R$, but also the ones which mix $\check{X}^R$ and $\hat{X}^m$. This in turn implies that restricted to such backgrounds, the string field theory action

[2]Such a representation of the string field theory action is known explicitly for the bosonic string theory [65, 67, 68]. It is also known for the bosonic sector of the heterotic string theory [69] which is sufficient for our argument.

will have an $O(d-1,1)\otimes O(d+p-1,1)$ symmetry, with the $O(d-1,1)$ transformations acting on the index l of $\psi_{l,\bar{l},a'}$, and the $O(d-1+p,1)$ transformations acting on the index $\bar{l}$ of $\psi_{l,\bar{l},a'}$.

We can bypass the use of string field theory by noting that the above arguments can also be used to establish an $O(d-1,1)\otimes O(d+p-1,1)$ symmetry of the S-matrix elements when the external states are restricted to carry zero momentum in d of the space-time directions and zero charge under p of the U(1) generators of the gauge group. Since the effective action of the theory is constructed from the S-matrix, the $O(d-1,1)\otimes O(d+p-1,1)$ symmetry must manifest itself as a symmetry of the effective action also. Note that this argument holds to all orders in the α' expansion, since nowhere we had to assume that the momenta carried by the external states in directions other than these d directions are small.

Let us now see how this transformation acts on some specific components of the string field. Let h_{MN}, b_{MN} and a_{MR} denote the components of the string field which couple to the graviton, the antisymmetric tensor and the gauge field vertex operators respectively. We shall choose normalizations such that $h_{mn}+b_{mn}$ couples to the vertex operator $c\bar{c}\partial\hat{X}^m\bar{\partial}\hat{X}^n e^{i\tilde{k}.\tilde{X}}$ and a_{mR} couples to the vertex operator $c\bar{c}\partial\hat{X}^m\bar{\partial}\check{X}^R e^{i\tilde{k}.\tilde{X}}$, where $\tilde{X}^\mu$ denote the set of coordinates other than $\hat{X}^m$. Let $\mathcal{S}$ and $\mathcal{R}$ be the $O(d-1,1)$ and $O(d-1+p,1)$ matrices associated with the Lorentz transformations involving the unbarred and the barred indices respectively. Then the $O(d-1,1)\otimes O(d+p-1,1)$ transformation acts on these fields as:

$$\left(\,(h'+b')\quad a'\,\right)=\mathcal{S}\left(\,(h+b)\quad a\,\right)\mathcal{R}^T \tag{3.4}$$

where $\left(\,(h+b)\quad a\,\right)$ is regarded as a $d\times(d+p)$ matrix. Similarly, if $(h_{m\mu}+b_{m\mu})$ couples to the vertex operator $c\bar{c}\partial\hat{X}^m\bar{\partial}\tilde{X}^\mu e^{i\tilde{k}.\tilde{X}}$, $(h_{\mu m}+b_{\mu m})$ couples to the vertex operator $c\bar{c}\partial\tilde{X}^\mu\bar{\partial}\hat{X}^m e^{i\tilde{k}.\tilde{X}}$, and $a_{\mu R}$ couples to the vertex operator $c\bar{c}\partial\tilde{X}^\mu\bar{\partial}\check{X}^R e^{i\tilde{k}.\tilde{X}}$, then these fields transform as,

$$\begin{aligned}
h'_{m\mu}+b'_{m\mu} &= \mathcal{S}_{mn}(h_{n\mu}+b_{n\mu})\\
h'_{\mu m}+b'_{\mu m} &= (h_{\mu n}+b_{\mu n})\mathcal{R}^T_{nm}+a_{\mu R}\mathcal{R}^T_{Rm}\\
a'_{\mu R} &= (h_{\mu n}+b_{\mu n})\mathcal{R}^T_{nR}+a_{\mu S}\mathcal{R}^T_{SR}
\end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) can be written in a more compact form in terms of a $D\times(D+p)$

dimensional matrix given by

$$\begin{pmatrix} (\mathbf{h}+\mathbf{b}) & \mathbf{a} \end{pmatrix} = \begin{pmatrix} (h+b)_{\mu\nu} & (h+b)_{\mu n} & a_{\mu R} \\ (h+b)_{m\nu} & (h+b)_{mn} & a_{mR} \end{pmatrix} \tag{3.6}$$

and take the form

$$\begin{pmatrix} (\mathbf{h}'+\mathbf{b}') & \mathbf{a}' \end{pmatrix} = S \begin{pmatrix} (\mathbf{h}+\mathbf{b}) & \mathbf{a} \end{pmatrix} R^T \tag{3.7}$$

Here, S and R are $D \times D$ and $(D+p) \times (D+p)$ dimensional matrices, respectively, and are given, in terms of $\mathcal{S}$ and $\mathcal{R}$, as

$$S = \begin{pmatrix} 1_{D-d} & 0 \\ 0 & \mathcal{S}_{mn} \end{pmatrix}, \quad R = \begin{pmatrix} 1_{D-d} & 0 & 0 \\ 0 & \mathcal{R}_{mn} & \mathcal{R}_{mS} \\ 0 & \mathcal{R}_{Rn} & \mathcal{R}_{RS} \end{pmatrix} \tag{3.8}$$

Similar transformation laws can be derived for other fields as well, but we shall not list them here. Note that the $O(d-1,1) \otimes O(d+p-1,1)$ 'symmetry' described above holds for any background that is independent of d of the space-time coordinates, and is neutral under p of the U(1) subgroups of the gauge group. This includes background massive fields as well. This completes our argument for the existence of a an $O(d-1,1) \otimes O(d+p-1,1)$ transformation in heterotic string theory.

3.3 $\mathrm{O(d, d+p)}$ SYMMETRY OF HETEROTIC STRING LOW-ENERGY EFFECTIVE ACTION

Although the general argument of the previous section guarantees the existence of an $O(d-1,1) \otimes O(d+p-1,1)$ symmetry in heterotic string theory, a realization of this symmetry in terms of fields that appear in the low-energy effective action is non-trivial. This is because the explicit relationship between the string field components h_{MN}, b_{MN}, a_{MR} and the fields that appear in the low-energy effective action is not known. In this section, we find a realization of the $O(d-1,1) \otimes O(d+p-1,1)$ transformations on the background fields in heterotic string theory, the dynamics of which is governed by the low-energy effective action (3.1). Our strategy is to write the action (3.1) in such a way that its symmetry becomes manifest.

We first impose the restrictions under which the $O(d-1,1) \otimes O(d+p-1,1)$symmetry is expected to appear in the low-energy theory. Let us split the coordinates X^M into two sets; $\hat{X}^m$ and $\tilde{X}^\mu$ where $1 \le m \le d$ and $1 \le \mu \le D-d$. In terms of the background fields, the restrictions of the previous section mean that all backgrounds are independent of $\hat{X}^m$. Furthermore, the gauge field background lies in a subgroup of the gauge group $E_8 \times E_8$ that commutes with p of the Cartan generators. More precisely, in the worldsheet theory, the elements of the Cartan subalgebra are the right moving $U(1)$ generators $\bar{\partial}X^I$ associated with the internal coordinates X^I of the heterotic string theory. Let us denote by $\check{X}^R$ $(1 \le R \le p)$ the internal coordinates corresponding to the $U(1)^p$ with which the gauge fields commute. Also, let $\tilde{a}$ denote the gauge group indices corresponding to the generators that commute with the $U(1)^p$ subgroup, and lie outside this subgroup. Then, explicitly, the allowed non-vanishing components of the gauge fields are $A_M^{\tilde{a}}$ and A_M^R.

To begin with, we shall further restrict to background field configurations for which $G_{m\mu} = B_{m\mu} = A_\mu^R = A_m^{\tilde{a}} = 0$; *i.e.*, to backgrounds of the form

$$G = \begin{pmatrix} \widetilde{G}_{\mu\nu} & 0 \\ 0 & \widehat{G}_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} \widetilde{B}_{\mu\nu} & 0 \\ 0 & \widehat{B}_{mn} \end{pmatrix}, \quad A_M^a = \begin{pmatrix} 0 & \widetilde{A}_\mu^{\tilde{a}} \\ \widehat{A}_m^R & 0 \end{pmatrix} \tag{3.9}$$

For block diagonal G and B, the gauge field is required to have the above form to insure the consistency of the equations of motion. For the above backgrounds, the equations of motion obtained by varying the action with respect to the field components that we have set to zero, are satisfied identically. Hence any invariance of the action for such restricted set of backgrounds will also imply invariance of the complete set of equations of motion. Afterwards we shall see how to write down the transformation laws in a more general case when such restrictions are not there. Now, we rewrite the low-energy effective action of the heterotic string theory given by equation (3.1), for the above restricted class of backgrounds. After an integration by parts, the action (3.1) can be shown to take the form:

$$\begin{aligned} S = -\int d^d\hat{X} \int d^{D-d}\tilde{X}\sqrt{\det \widetilde{G}}e^{-\chi}\Big(& -\frac{1}{32}\widetilde{G}^{\mu\nu}Tr(\tilde{\partial}_\mu ML\tilde{\partial}_\nu ML) \\ & +\Lambda - \widetilde{G}^{\mu\nu}\tilde{\partial}_\mu\chi\tilde{\partial}_\nu\chi - \widetilde{R}^{(D-d)}(\widetilde{G}) + \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + \frac{1}{8}\sum_{\tilde{a}}\widetilde{F}_{\mu\nu}^{\tilde{a}}\widetilde{F}^{\tilde{a}\mu\nu}\Big) \end{aligned} \tag{3.10}$$

where,

$$L = \begin{pmatrix} \eta_d & 0 \\ 0 & -\eta_{d+p} \end{pmatrix} \tag{3.11}$$

and

$$\chi = \Phi - \ln\sqrt{-\det \widehat{G}} \tag{3.12}$$

The matrix M appearing in the action is a $(2d+p)\times(2d+p)$ dimensional matrix given by

$$M = \begin{pmatrix} (K^T-\eta_d)\widehat{G}^{-1}(K-\eta_d) & (K^T-\eta_d)\widehat{G}^{-1}(K+\eta_d) & -(K^T-\eta_d)\widehat{G}^{-1}\widehat{A} \\ (K^T+\eta_d)\widehat{G}^{-1}(K-\eta_d) & (K^T+\eta_d)\widehat{G}^{-1}(K+\eta_d) & -(K^T+\eta_d)\widehat{G}^{-1}\widehat{A} \\ -\widehat{A}^T\widehat{G}^{-1}(K-\eta_d) & -\widehat{A}^T\widehat{G}^{-1}(K+\eta_d) & \widehat{A}^T\widehat{G}^{-1}\widehat{A} \end{pmatrix} \tag{3.13}$$

Here η_d denotes the d dimensional Minkowski metric $diag(-1,1,\ldots 1)$, $\widehat{A}$ is the matrix $\widehat{A}_{mR} \equiv \widehat{A}^R_m$, and K is given by

$$K = -\widehat{B} - \widehat{G} - (1/4)\widehat{A}\widehat{A}^T \tag{3.14}$$

The action (3.10) is manifestly invariant under the transformations

$$M \to M' = \Omega M \Omega^T \tag{3.15}$$

$$\chi \to \chi, \quad \widetilde{G}_{\mu\nu} \to \widetilde{G}_{\mu\nu}, \quad \widetilde{B}_{\mu\nu} \to \widetilde{B}_{\mu\nu}, \quad \widetilde{A}^{\tilde{a}}_\mu \to \widetilde{A}^{\tilde{a}}_\mu \tag{3.16}$$

where,

$$\Omega = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{R} \end{pmatrix} \tag{3.17}$$

with $\mathcal{S}$ and $\mathcal{R}$ being $O(d-1,1)$ and $O(d+p-1,1)$ matrices respectively. At the linearized level, $\widehat{G}_{mn} = \eta_{mn} + h_{mn}$, $\widehat{B}_{mn} = b_{mn}$ and $\widehat{A}^R_m = a_{mR}$, and the transformations given above agree with the transformations of h, b and a given in equation (3.4). The transformation law of Φ can also be shown to agree with the linearized transformations [16]. This proves the $O(d-1,1)\otimes O(d+p-1,1)$ symmetry of heterotic string low-energy effective action when the background fields are restricted as described above. The action of the transformations on the background fields is uniquely determined by the equations (3.15) and (3.16). Note that the symmetry group of the action (3.10) acting on the matrix M is actually larger than $O(d-1,1)\otimes O(d+p-1,1)$. In fact, the action is invariant under any transformation Ω which keeps the metric L in (3.11) invariant: $\Omega L \Omega^T = L$. Clearly, these transformations generate the group $O(d,d+p)$. However, as in the non-heterotic case, all transformations outside the $O(d-1,1)\otimes O(d+p-1,1)$ subgroup can be shown to generate d dimensional general linear coordinate transformations, and gauge transformations of the antisymmetric tensor field and the gauge field which are trivial symmetries of the theory.

One way to derive the transformation laws given in equation (3.15) under the $O(d-1,1)\otimes O(d+p-1,1)$ transformation is as follows. Let us imagine for the time being that all the dimensions associated with the coordinates $\hat{X}^m$ have been compactified (the effective action does not depend on whether these dimensions are compact or not). In that case the low energy effective field theory involving the moduli of this compact space (together with the moduli associated with the internal coordinates) is governed by the Zamolodchikov metric for these moduli, which, in turn, is invariant under the $O(d,d+p)$ group introduced in refs.[7, 8]. Thus the action of the $O(d-1,1)\otimes O(d+p-1,1)$ group on the various fields may be obtained from the action of this $O(d,d+p)$ group on the moduli space. This action, in turn, may be read out directly from the analysis of reference [95, 109] and is given by, $M\to\Omega M\Omega^T$ where M is the same matrix as given in equation (3.13) and Ω is an $O(d,d+p)$ matrix which preserves the matrix diag$(\eta_d,-\eta_{d+p})$.

Let us now consider the case where $G_{\mu m}$, $B_{\mu m}$ and $A_{\mu R}$ are non zero. However, for simplicity, we will now assume that the background gauge field is Abelian, and belongs to the $U(1)^{16}$ subgroup of the gauge group. Since this subgroup commutes with all the 16 U(1) generators, we can take $p=16$. In this case, we shall define a $(2D+16)\times(2D+16)$ matrix $\mathcal{M}$ as:

$$\mathcal{M}=\begin{pmatrix} (\mathcal{K}^T-\eta_D)G^{-1}(\mathcal{K}-\eta_D) & (\mathcal{K}^T-\eta_D)G^{-1}(\mathcal{K}+\eta_D) & -(\mathcal{K}^T-\eta_D)G^{-1}A \\ (\mathcal{K}^T+\eta_D)G^{-1}(\mathcal{K}-\eta_D) & (\mathcal{K}^T+\eta_D)G^{-1}(\mathcal{K}+\eta_D) & -(\mathcal{K}^T+\eta_D)G^{-1}A \\ -A^TG^{-1}(\mathcal{K}-\eta_D) & -A^TG^{-1}(\mathcal{K}+\eta_D) & A^TG^{-1}A \end{pmatrix} \tag{3.18}$$

where,

$$\mathcal{K}=-B-G-(1/4)AA^T \tag{3.19}$$

The gauge index of A now runs over all the 16 internal coordinates and the space-time index runs over all the D coordinates. With these general background field configurations, the low-energy effective action (3.1) can be written as

$$S = -\int d^d\hat{X} \int d^{D-d}\tilde{X} e^{-\psi} \Big(\Lambda - G^{\mu\nu}\tilde{\partial}_\mu\psi\tilde{\partial}_\nu\psi - \frac{1}{32} G^{\mu\nu} Tr(\tilde{\partial}_\mu \mathcal{M}\mathcal{L}\tilde{\partial}_\nu\mathcal{M}\mathcal{L}) + \tilde{\partial}_\mu\psi\tilde{\partial}_\nu G^{\mu\nu} - \frac{1}{2} P^{\nu\mu}_{(\mu\nu)} \Big) \tag{3.20}$$

where,

$$\mathcal{L} = \begin{pmatrix} \eta_D & 0 \\ 0 & -\eta_{D+16} \end{pmatrix} \tag{3.21}$$

and,

$$\psi = \Phi - \ln\sqrt{-\det G} \tag{3.22}$$

The term $P^{\nu\mu}_{(\mu\nu)}$ in the action can be constructed in terms of the matrix $\mathcal{M}$ as follows. We first define a matrix V as

$$V = \begin{pmatrix} \eta_D/\sqrt{2} & -\eta_D/\sqrt{2} & 0 \\ 1_D/\sqrt{2} & 1_D/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

Using this matrix, we transform $\mathcal{M}$ and $\mathcal{L}$ to

$$\check{\mathcal{M}} = \frac{1}{2} V\mathcal{M}V^T = \begin{pmatrix} G^{-1} & -G^{-1}\mathcal{K} & G^{-1}A/\sqrt{2} \\ -\mathcal{K}^T G^{-1} & \mathcal{K}^T G^{-1}\mathcal{K} & -\mathcal{K}^T G^{-1} A/\sqrt{2} \\ A^T G^{-1}/\sqrt{2} & -A^T G^{-1}\mathcal{K}/\sqrt{2} & A^T G^{-1} A/2 \end{pmatrix} \tag{3.24}$$

and

$$\check{\mathcal{L}} = V\mathcal{L}V^T = \begin{pmatrix} 0 & 1_D & 0 \\ 1_D & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix} \tag{3.25}$$

We now define the matrices $P_{(\mu\nu)}, \ldots, Z_{(\mu\nu)}$ in terms of $\check{\mathcal{M}}$ and $\check{\mathcal{L}}$ through the relations:

$$(\partial_\mu \check{\mathcal{M}}\check{\mathcal{L}}(\check{\mathcal{M}} - \check{\mathcal{L}})\check{\mathcal{L}}\partial_\nu\check{\mathcal{M}}) = \begin{pmatrix} P_{(\mu\nu)} & Q_{(\mu\nu)} & R_{(\mu\nu)} \\ S_{(\mu\nu)} & T_{(\mu\nu)} & W_{(\mu\nu)} \\ X_{(\mu\nu)} & Y_{(\mu\nu)} & Z_{(\mu\nu)} \end{pmatrix} \tag{3.26}$$

In the above, $P_{(\mu\nu)}$ is a $D \times D$ matrix. The term $P^{\rho\sigma}_{(\mu\nu)}$ appearing in the effective action (3.20) is now defined to be the $\rho\sigma$ component of this matrix. The advantage of choosing this representation for $P_{(\mu\nu)}$ is that its transformation properties are now easily obtainable from those of $\mathcal{M}$

The action given in (3.20) can be shown to be invariant under a transformation of the form

$$\mathcal{M} \to \Omega\, \mathcal{M}\, \Omega^T, \qquad \psi \to \psi \tag{3.27}$$

with,

$$\Omega = \begin{pmatrix} 1_{D-d} & & & \\ & \mathcal{S} & & \\ & & 1_{D-d} & \\ & & & \mathcal{R} \end{pmatrix} \tag{3.28}$$

where $\mathcal{S}$ and $\mathcal{R}$ are the $O(d-1,1)$ and $O(d+p-1,1)$ matrices discussed previously (with $p = 16$). The invariance of the actiond under the above transformation follows from the fact that $G^{\mu\nu}$ and $P^{\rho\sigma}_{(\mu\nu)}$ remain invariant under these transformations. Note that when $G_{m\mu}$, $B_{m\mu}$ and $A_{\mu R}$ are zero, the transformation laws given by equation (3.27) are identical to those given in (3.15). Also, these transformations reduce to the ones given in equations (3.4) and (3.5) when $G_{MN} - \eta_{MN}$, B_{MN} and A_{MR} are small and hence can be identified with h_{MN}, b_{MN} and a_{MR} respectively. Also note that if we set the gauge field to zero, then we obtain the $O(d,d)$ transformations of G and B which are not necessarily block diagonal. This, as promised in Chapter 1, generalizes the $O(d,d)$ transformations to the more general class of backgrounds which are not constrained to be of the form given in (1.2). We will use this form of the transformation in the next chapter.

Before we conclude this section, let us remark that although the general arguments of section 2 guarantee the existence of an $O(d-1,1) \otimes O(d+p-1,1)$ 'symmetry' of the string theory for appropriate backgrounds to all orders in α', it does not guarantee that the transformation laws, when expressed in terms of the fields G_{MN}, B_{MN},A^a_M and Φ will remain unchanged when we include corrections that are higher order in α'. This is due to the fact that the functional relationship between the string fields and the fields that appear in the effective field theory may undergo modification when we include the effect of higher derivative terms. Evidence of such modification in the transformation laws has already been observed [64, 53, 16]. In [28, 29] a procedure has been suggested for obtaining the exact transformations of the metric and the dilaton field but not of the antisymmetric tensor field in the non-heterotic cases.

3.4 APPLICATIONS OF $\mathrm{O(d, d+p)}$ TRANSFORMATIONS

In this section we will apply the above transformations to two known solutions of heterotic string theory to generate new solutions. The analysis shows that the transformation parameters are related to physical quantities like the electric charge and the antisymmetric tensor field charge of the solution. The new solutions have non-trivial space-time structure which depend on the transformation parameters.

First, we take our starting solution to be the black six brane solution of reference[77] carrying a magnetic charge. (For related work see [76]-[94]) The solution is given by the following form of the metric and other fields

$$ds^2 = -\frac{(1-r_+/r)}{(1-r_-/r)}dt^2 + \frac{dr^2}{(1-r_+/r)(1-r_-/r)} + r^2 d\Omega_2^2 + \sum_{i=1}^{6} dX^i dX^i \tag{3.29}$$

$$\Phi = -\ln(1-r_-/r) + \Phi_0 \tag{3.30}$$

$$F^1 = 2\sqrt{2} Q_M \epsilon_2 \tag{3.31}$$

where, $d\Omega_2$ is the line element on a two sphere, and ϵ_2 is the volume form on the same two sphere. Φ_0, r_+ and r_- (with $r_+ > r_-$) are the three independent parameters labeling the solution, and Q_M is the (quantized) magnetic charge carried by the black hole, and is given by,

$$Q_M = \sqrt{r_+ r_-/2} \tag{3.32}$$

For definiteness, we have taken the magnetic field to lie in the $U(1)$ subgroup generated by the first internal coordinate. We shall now perform the $O(d-1,1) \otimes O(d+p-1,1)$ transformation on this solution to generate new solutions. To this end, note that the solution is independent of the coordinate t and also the six coordinates X^i, thus here $d = 7$. Furthermore, the presence of the magnetic field requires A^1 to have a non-vanishing component tangent to the 2-sphere, thus if we want to satisfy the condition $A_{\mu R} = 0$, we must exclude the direction 1 from the set of directions R. Although we have shown that this is not necessary, we shall first consider this case. Thus here $p = 15$. Although we can involve all the 7 space-time coordinates, and all the 15 internal coordinates in the transformation, a general transformation of this kind will generate solutions which will be related by rotation in the external and/or internal space. It is easy to see this for the transformations involving the coordinates X^i. For gauge fields belonging to $E_8 \times E_8$, rotations among the 15 internal coordinates can generate inequivalent field configurations since $O(16)$ is not a subgroup of the gauge group. But this only changes the direction of the gauge field in the final solution without modifying the essential properties of the solution. Thus, a set of inequivalent field configurations is generated by taking appropriate $O(d-1,1) \otimes O(d+p-1,1)$transformations involving the coordinate t, one of the space coordinates (say X^1) and one of the internal coordinates (say $\check{X}^2$). The symmetry group in this case is $O(1,1) \otimes O(2,1)$. The diagonal $O(1,1)$ subgroup contained in this symmetry group corresponds to a Lorentz transformation of the solution in the $t - X^1$ space. We may fix a Lorentz frame by choosing the matrix $\mathcal{S}$ to be the identity matrix. Thus we are left with the $O(2,1)$ matrix $\mathcal{R}$ parametrized by the three Euler angles. A further reduction of the parameters may be made by noting that the original solution is left invariant if we choose $\mathcal{R}$ to be a rotation in the $X^1 - \check{X}^2$ plane. Thus the general solution is obtained by taking $\mathcal{R}$ to be a boost in the $t - \check{X}^2$ direction followed by a boost in the $t - X^1$ direction:

$$\mathcal{R} = \begin{pmatrix} \cosh\alpha_2 & \sinh\alpha_2 & 0 \\ \sinh\alpha_2 & \cosh\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\alpha_1 & 0 & \sinh\alpha_1 \\ 0 & 1 & 0 \\ \sinh\alpha_1 & 0 & \cosh\alpha_1 \end{pmatrix} \tag{3.33}$$

We can now calculate the transformed solution in a straightforward way using equations (3.15), (3.16). In terms of the new set of parameters γ, β and r_0 which are related to the transformation parameters by

$$\begin{aligned} \gamma &= \sinh\alpha_1 \\ \beta &= \cosh\alpha_1 \sinh\alpha_2 \\ r_0 &= \tfrac{1}{2}((r_+ + r_-) - (r_+ - r_-)\sqrt{1+\beta^2+\gamma^2}) \end{aligned} \tag{3.34}$$

the new background, fields corresponding to the transformed solution, are given by

$$
\begin{aligned}
ds^2 &= -\frac{1}{4(r-r_0)^2}(4(r-r_+)(r-r_-)-(r_+-r_-)^2\beta^2)dt^2+\beta\frac{r_+-r_-}{(r-r_0)}dX^1dt \\
&\quad +\sum_{i=1}^{6}dX^idX^i+\frac{dr^2}{(1-r_+/r)(1-r_-/r)}+r^2d\Omega_2^2 \\
B_{t1} &= \beta\frac{r_+-r_-}{2(r-r_0)} \\
A_t^2 &= \gamma\frac{r_+-r_-}{(r-r_0)} \\
A_1^2 &= 0 \\
F^1 &= 2\sqrt{2}Q_M\epsilon_2 \\
\Phi &= -\ln(1-r_0/r)+\Phi_0
\end{aligned}
\tag{3.35}
$$

This solution is characterized by an electric field strength as well as an antisymmetric tensor field strength, given by

$$
\begin{aligned}
F_{rt}^2 &= \partial_r A_t^2 = -\gamma\frac{(r_+-r_-)}{(r-r_0)^2} \\
H_{rt1} &= \partial_r B_{t1} = -\beta\frac{(r_+-r_-)}{2(r-r_0)^2}
\end{aligned}
\tag{3.36}
$$

Hence, besides carrying a magnetic charge, the new solution also carries both electric and antisymmetric tensor gauge field charges, proportional to the parameters γ and β respectively.

Let us now discuss singularities of the new solution (3.35). It can be easily seen that the matrix

$$
\begin{pmatrix} G_{tt} & G_{t1} \\ G_{t1} & G_{11} \end{pmatrix}
$$

has zero eigenvalues at $r=r_+$ and $r=r_-$. The component G_{rr} has poles at precisely these values of r, as can be seen from equation (3.29). These singularities represent coordinate singularities, and can be removed by appropriate coordinate choice. To see this, let us first define new coordinates t', X' and ρ through the relations:

$$\begin{aligned} t &= t'\cosh\theta - X'\sinh\theta \\ X^1 &= X'\cosh\theta - t'\sinh\theta \\ r &= \rho + r_+ \end{aligned} \tag{3.37}$$

where,

$$\tanh\theta = \frac{\beta}{2}\frac{r_+ - r_-}{r_+ - r_0} \tag{3.38}$$

In this coordinate system, the metric near $r = r_+$ takes the form:

$$\begin{aligned} ds^2 = & -\frac{2\cosh\theta\sinh\theta}{\beta(r_+ - r_0)}\rho(dt')^2(1+\mathcal{O}(\rho)) \\ & +\frac{2(r_+ - r_0)}{\beta(r_+ - r_-)}\Big[1 - \frac{\beta^2(r_+ - r_-)^2}{4(r_+ - r_0)^2}\Big]^2\cosh\theta\sinh\theta(dX')^2(1+\mathcal{O}(\rho)) \\ & -\frac{2\rho}{r_+ - r_0}\Big[1 - \frac{1}{4}\frac{\beta^2(r_+ - r_-)^2}{(r_+ - r_0)^2} - \frac{r_+ - r_-}{r_+ - r_0}\Big]\cosh\theta\sinh\theta dX'dt'(1+\mathcal{O}(\rho)) \\ & +\frac{(r_+)^2}{(r_+ - r_-)\rho}(d\rho)^2(1+\mathcal{O}(\rho)) + (r_+ + \rho)^2(d\Omega_2)^2 + \sum_{i=2}^{6} dX^i dX^i \end{aligned} \tag{3.39}$$

From this we see that the metric has a singularity at $\rho = 0$. This singularity is removed by defining new coordinates u, v as,

$$u = \sqrt{\rho}e^{at'}, \quad v = \sqrt{\rho}e^{-at'} \tag{3.40}$$

where,

$$a = \sqrt{\frac{\cosh\theta\sinh\theta(r_+ - r_-)}{2(r_+)^2(r_+ - r_0)\beta}} \tag{3.41}$$

In this coordinate system the metric takes the form:

$$\begin{aligned} ds^2 = \quad & \frac{4(r_+)^2}{(r_+ - r_-)}dudv + \frac{2(r_+ - r_0)}{\beta(r_+ - r_-)}\Big[1 - \frac{\beta^2(r_+ - r_-)^2}{4(r_+ - r_0)^2}\Big]^2\cosh\theta\sinh\theta(dX')^2 \\ & +(r_+)^2(d\Omega_2)^2 + \sum_{i=2}^{6} dX^i dX^i + \mathcal{O}(u,v) \end{aligned} \tag{3.42}$$

From this we see that the metric is non-singular in this coordinate system at $u = 0$ or $v = 0$. It can also be seen easily that both the electric and the antisymmetric tensor fields are non-singular at $r = r_+$ in the new coordinate system. Similar change of coordinates can also be carried out near the singularity at $r = r_-$ to show that this also represents a coordinate singularity. In this case we again look for a coordinate transformation of the form

$$
\begin{aligned}
t &= t'' \cosh\phi - X'' \sinh\phi \\
X^1 &= X'' \cosh\phi - t'' \sinh\phi \\
r &= r_- + \rho'
\end{aligned}
\tag{3.43}
$$

as in equation (3.37), so as to bring the metric in the standard form near the singular surface at $r = r_-$. It can be easily seen that if $|\beta| < \gamma^2/2$, then it is possible to find a ϕ given by $\tanh\phi = \beta(r_+ - r_-)/2(r_- - r_0)$ for which $G_{t''t''}$ and $G_{X''t''}$ are of order ρ', and $G_{X''X''}$ is of order 1 as $r \to r_-$. On the other hand, if $|\beta| > \gamma^2/2$, then it is possible to find a ϕ given by $\coth\phi = \beta(r_+ - r_-)/2(r_- - r_0)$, for which $G_{X''X''}$ and $G_{X''t''}$ are of order ρ' and $G_{t''t''}$ is of order unity as $r \to r_-$. Thus the global structure of the solution resembles that of a Reissner-Nordstrom black hole in the first case, and that of the black string solution of [94, 17] in the second case. The above results are summarized in the following table

	G_{tt}	G_{rr}	G_{XX}
$r > r_+$:	$-$	$+$	$+$
$r_+ > r > r_-$:	$+$	$-$	$+$
$r < r_-\ , \|\beta\| < \gamma^2/2$:	$-$	$+$	$+$
$r < r_-\ , \|\beta\| > \gamma^2/2$:	$+$	$+$	$-$

where, the components of the metric refer to the coordinate systems (3.37),(3.43) near the surfaces $r = r_\pm$. On the other hand, the point $r = r_0$, as well as $r = 0$, represents genuine singularities of the solution. ($\Phi \to \pm\infty$ near these points.) Since for real β and γ, $r_0 \le r_\pm$, we see that the solution represents a genuine singularity surrounded by two horizons. Solvable conformal field theories corresponding to black string solutions with two horizons have been found previously by Horne and Horowitz [94].

Note that if we take $\beta = 0$, then the solution represents the direct product of a four dimensional black hole carrying magnetic and electric charge, and a six dimensional flat space described by the coordinates X^i ($1 \leq i \leq 6$). If we compactify the coordinates X^i (say on a Calabi-Yau manifold, or a six dimensional torus), the result would be a four dimensional black hole carrying electric and magnetic charge. The full solution, on the other hand, may be regarded as a black string in 5 dimensions by compactifying the coordinates $X^2, \ldots X^6$. The two gauge field charges lie in different U(1) subgroups of the gauge group. In this respect, these solutions are different from the ones discussed in reference [93] in which the electric and the magnetic charges lie in the same U(1) subgroups of the gauge group. On the other hand, these solutions can be identified to the black hole solutions of Gibbons and Maeda [87] carrying electric and magnetic charge, if we interpret the electric and magnetic charge in their solution to belong to different U(1) subgroups of the gauge group. Note that this is the only way to interpret the solutions of Gibbons and Maeda in the context of string theory, since if the electric and the magnetic fields belong to the same U(1) subgroup, we need to take into account the effect of the gauge Chern-Simons term coupling to the antisymmetric tensor gauge field strength, which was not included in the analysis of reference [87]. If we further set the magnetic charge Q_M to zero, the solution reduces to the charged black hole solution of reference [79]. (Note that the metric $\hat{ds}^2$ considered in references [87, 79] is related to the metric ds^2 given in equation (3.35) through the relation $\hat{ds}^2 = e^{-\Phi} ds^2$[77].)

Since we have derived the transformation laws of various fields under $O(d-1,1) \otimes O(d+p-1,1)$ transformations even when $G_{m\mu}$, $B_{m\mu}$ and $A_{\mu R}$ are non zero, we could in principle perform an $O(d-1,1) \otimes O(d+p-1,1)$ rotation that includes the 1 direction of the gauge field. Note, however, that in this case the initial gauge potential corresponds to a magnetic monopole configuration. The gauge potential, therefore, has a singularity and is not globally well defined. In such a situation, the gauge field can be defined in separate coordinate patches in such a way that, on the overlap region, the fields in different patches are related by a gauge transformation. This, in general, implies that the transformed fields also need to be defined in separate coordinate patches, and are related by gauge and general coordinate transformation on the overlap. To see this let us consider the transformation of the fields in the asymptotic region $r \to \infty$, so that $G_{\mu\nu} - \eta_{\mu\nu}$, $B_{\mu\nu}$ and $A_{\mu R}$ are small. If we choose

$S=1$, and R to be a $O(1,1)$ transformation that mixes the t coordinate with the 1 direction in the internal space, the transformed fields take the form:

$$\begin{aligned} G'_{\alpha t} &= \tfrac{1}{2}\sinh\theta A^1_\alpha \\ B'_{\alpha t} &= \tfrac{1}{2}\sinh\theta A^1_\alpha \\ A^{1\prime}_\alpha &= A^1_\alpha \cosh\theta \end{aligned} \tag{3.44}$$

where α denotes any of the three directions x, y or z on which the original solution depends. Let A^1_α and $\bar{A}^1_\alpha$ be the components of the original gauge field in the two different coordinate patches, then $A^1_\alpha - \bar{A}^1_\alpha = \partial_\alpha\Lambda$, where Λ is a function which is not single valued under a 2π rotation about the z axis (although $e^{ie\Lambda}$ is). From equation (3.44) we see that $G'_{\alpha t} - \bar{G}'_{\alpha t}$ is now given by $(1/2)\sinh\theta\partial_\alpha\Lambda$, where G' and $\bar{G}'$ denote the transformed metric in the two coordinate patches. This shows that G' and $\bar{G}'$ are related by a coordinate transformation of the form $t \to t + \Lambda\sinh\theta$. However, since Λ is not a single valued function of the coordinates, this coordinate transformation is not globally well defined.

We could also have started with the metric which represents black holes carrying quantized antisymmetric tensor gauge field charge, instead of magnetic charge. This solution is given by [77]:

$$\begin{aligned} ds^2 &= -\frac{1-r_+^2/r^2}{1-r_-^2/r^2}dt^2 + \frac{dr^2}{(1-r_+^2/r^2)(1-r_-^2/r^2)} + r^2 d\Omega_3^2 + \sum_{i=1}^{5} dX^i dX^i \\ \Phi &= -\ln(1-r_-^2/r^2) + \Phi_0 \\ \widetilde{H}_{\alpha\beta\gamma} &= Q(\epsilon_3)_{\alpha\beta\gamma} \end{aligned} \tag{3.45}$$

where $d\Omega_3$ is the line element on a three sphere, ϵ_3 is the volume form on the same three sphere, and $Q = r_+ r_-$. In this solution, the expressions for G_{tt} and Φ are similar to those given in equations(3.29) and (3.30), except that the ratios $r/r_\pm$ are replaced by $(r/r_\pm)^2$. As a result, the final transformed solution will have the same form as given in equations(3.35) and (3.34) with r, r_0 and $r_\pm$ replaced by r^2, $(r_0)^2$ and $(r_\pm)^2$ everywhere in the expression for

$\hat{G}$, $\hat{B}$ and $\hat{A}$. Thus the final solution will take the form:

$$
\begin{aligned}
ds^2 &= -\frac{1}{4(r^2-r_0^2)^2}(4(r^2-r_+^2)(r^2-r_-^2)-(r_+^2-r_-^2)\beta^2)dt^2+\beta\frac{r_+^2-r_-^2}{r^2-r_0^2}dX^1dt \\
&\quad +\sum_{i=1}^{5}dX^idX^i+\frac{dr^2}{(1-r_+^2/r^2)(1-r_-^2/r^2)}+r^2d\Omega_3^2 \\
\hat{B}_{t1} &= \beta\frac{r_+^2-r_-^2}{2(r^2-r_0^2)} \\
A_t &= \gamma\frac{r_+^2-r_-^2}{r^2-r_0^2} \\
A_1 &= 0 \\
\Phi &= -\ln(1-r_0^2/r^2) \\
\tilde{H}_{\alpha\beta\gamma} &= Q(\epsilon_3)_{\alpha\beta\gamma}
\end{aligned}
\tag{3.46}
$$

where,

$$
r_0^2=\frac{1}{2}((r_+^2+r_-^2)-(r_+^2-r_-^2)\sqrt{1+\beta^2+\gamma^2})
\tag{3.47}
$$

Note that in this case the antisymmetric tensor gauge field has a 'magnetic' type component denoted by $\tilde{H}_{\alpha\beta\gamma}$ and also an electric type component denoted by $\hat{H}_{rt1}\equiv\partial_r\hat{B}_{t1}$. Again, by taking the directions $X^2,\ldots X^5$ to be compact, this solution may be regarded as a black string solution in six dimensions.

3.5 SUMMARY

In this chapter we have shown that when the background fields in heterotic string theory are restricted to configurations which are independent of d of the space-time coordinates and when the gauge field lies in a subgroup which commutes with p of the Cartan generators of the gauge group, then the low energy effective action has an $O(d-1,1)\otimes O(d+p-1,1)$ symmetry. This group acts non-linearly on the background fields and, after the inclusion of some exact gauge symmetries, can be completed to the $O(d,d+p)$ group. We have also discussed the string theory origins of these transformations which guarantee their existence to all orders in the σ-model perturbation theory. Given a classical solution of the heterotic string theory satisfying the above mentioned restrictions, we can generate other classical solutions by applying an $O(d-1,1) \otimes O(d+p-1,1)$ transformation on the original solution. By using these transformations on the known black 6-brane solution of the heterotic string theory carrying magnetic charge, we have generated new solutions carrying magnetic, electric and antisymmtric tensor gauge field charge. These solutions are labeled by four continuous and one discrete parameters, characterizing the mass, the electric charge, the antisymmetric tensor gauge field charge, the asymptotic value of the dilaton field, and the magnetic charge of the 6-brane respectively. By compactifying 5 of the directions this solution may be regarded as a black string solution in five dimensions. Using this method we have also constructed black string solutions in six dimensions carrying electric charge, and both, electric and magnetic type antisymmetric tensor gauge field charges.

Chapter 4

THE O(d, d) AND O(d, d + 16) DEFORMATIONS AND EXTENDED SUPERSYMMETRY ON THE WORLDSHEET

4.1 INTRODUCTION

In this chapter, based on the work appearing in [25], we will show that the $O(d,d;\mathrm{R})$ deformations of the superstring vacua and the $O(d,d+16;\mathrm{R})$ deformations of the heterotic string vacua preserve extended worldsheet supersymmetry and, hence, generate superconformal deformations. The transformations of the complex structures are given and the action of the discrete duality subgroup is discussed. The analysis is performed by rewriting the transformations in terms of the metric vielbein and by introducing certain space-time connections induced due to the non-linear action of the transformations on the background fields.

String theories with extended supersymmetry on the worldsheet are important because of the restrictions which the extra global supersymmetries impose on the target space geometry [97]-[105] (for a review see [106]), as well as their natural relation to space-time supersymmetry [104, 107, 108] . The study of N=2 theories, and the profound structures associated with them has been an area of intense investigation in recent years. We start our discussion in this section with a brief introduction to non-linear σ-models with extended supersymmetry on the worldsheet and then consider the issue of their compatibility with $O(d,d)$ and $O(d,d+16)$ transformations. The superstring and heterotic string theories have, by construction, local $(1,1)$ and $(1,0)$ supersymmetries on the worldsheet respectively. The local worldsheet supersymmetry is preserved in the presence of general background fields though the explicit supersymmetry transformation equations get modified. For certain configurations of the background fields, however, the theory is invariant under a larger class of supersymmetry transformations. In such a situation, when the supersymmetry group is enlarged without enlarging the field content of the theory beyond what is required for the original local supersymmetry, the theory is said to have extended supersymmetry. In non-linear σ-models the presence of each extended supersymmetry, in each one of the chiral sectors in the worldsheet theory, is found to be related to the existence of a complex structure J on the target space manifold. For example, a non-linear σ-model with local $N = 1$ supersymmetry, in both the left and right moving sectors, can have a second supersymmetry in both sectors provided the target manifold admits two complex structures J^+ and J^- [97]-[106]. The relationship between extended supersymmetries and complex structures arises as follows: Consider, for definiteness, the superstring theory with bosonic coordinates X^M and their fermionic superpartners ${\psi_\mp}^M$, where, $\pm$ denote the fermion chiralities on the worldsheet. This theory has $N = 1$ supersymmetry both in the left moving and the right moving sectors. Under a supersymmetry transformation, the variation of X^M is given in terms of ${\psi_\pm}^M$ and *vice versa.* The $\pm$ sign indicate whether the supersymmetry is in the left-moving or in the right-moving sector. Now consider a new set of supersymmetry transformations which are obtained from the previous ones on replacing ${\psi_\pm}^M$ by ${J^\pm}^M_N{\psi_\pm}^N$, where, ${J^\pm}^M_N$ are linear maps on the tangent space of the target manifold. The requirements that the worldsheet theory is invariant under the new supersymmetry transformations and that the old and new supersymmetries are independent of each other, lead to some conditions on ${J^\pm}^M_N$ which, in turn, imply some mathematically well defined constraints on the geometry on the target manifold. The conditions on ${J^\pm}^M_N$ can be written as follows:

$$ {J^\pm}^M_N {J^\pm}^N_K = -\delta^M_K \tag{4.1} $$

$$ {N^\pm}^K_{MN} \equiv {J^\pm}^L_M \partial_{[L} {J^\pm}^K_{N]} - {J^\pm}^L_N \partial_{[L} {J^\pm}^K_{M]} = 0 \tag{4.2} $$

$$ {J^\pm}^M_K G_{MN} {J^\pm}^N_L = G_{KL} \tag{4.3} $$

$$ \nabla^\pm_M {J^\pm}^N_K \equiv \partial_M {J^\pm}^N_K + {\Omega^\pm}^N_{ML} {J^\pm}^L_K - {\Omega^\pm}^L_{MK} {J^\pm}^N_L = 0 \tag{4.4} $$

The $\Omega^\pm$ in the last equation are non-Riemannian connections constructed in terms of the

Christoffel symbol and the torsion tensor as

$$
\begin{aligned}
\Omega^{\pm N}_{\ \ ML} &= \Gamma^{N}_{ML} \pm G^{NP}\, H_{PML} \\
&= \frac{1}{2} G^{NP} \left[\partial_M (G \mp B)_{PL} + \partial_L (G \mp B)_{MP} - \partial_P (G \mp B)_{ML}\right] \qquad (4.5)
\end{aligned}
$$

If the above conditions are satisfied, the theory has extended $(2,2)$ supersymmetry. These are the most general conditions for the existence of (2,2) extended supersymmetry on the worldsheet [105].

We will now briefly discuss the geometric significance of the above equations. Equation (4.1) means that the target space manifold admits two almost complex structures J^+ and J^-. The existence of a complex structure implies that the manifold is even dimensional. Equation (4.2) is the vanishing condition for the Nijenhuis tensors $N^{\pm K}_{\ \ MN}$ associated with the complex structures $J^\pm$ and implies the integrability of the complex structures. For a given complex structure J, this means that we can choose complex coordinates $z^a, \bar{z}^{\bar{a}}$ such that J is diagonal over a coordinate patch ($J^a_b = i\delta^a_b, J^{\bar{a}}_{\bar{b}} = -i\delta^{\bar{a}}_{\bar{b}}, J^{\bar{a}}_b = J^a_{\bar{b}} = 0$) and the transition functions are holomorphic in z^a. Equation (4.3) is the condition that the manifold admits a metric which is hermitian with respect to both of the complex structures. Equation (4.4) is the condition of covariant constancy of the complex structures with respect to the generalized connections given in (4.5). In the special case when the torsion is zero, this equation is very restrictive and defines a Kähler manifold. Note that in this case the two complex structures can be chosen to be the same. The form of the torsion H_{MNP} has important implications for superfield representation of theories with extended $N = 2$ supersymmetry. Theories defined on a Kähler manifold have a manifest supersymmetric representation in terms of chiral superfields [98, 99]. A more general situation arises when the torsion is not zero, but the complex structures, J^+ and J^-, commute with each other. These theories can be written in manifestly supersymmetric form in terms of the chiral and twisted chiral superfields [100, 102, 103]. When the two complex structures do not commute, a superfield representation is not known in general.

Theories with extended $N = 2$ supersymmetry on the worldsheet can be further restricted, giving rise to $N = 4$ extended supersymmetry. In this case, the target manifold must admit three complex structures in each chiral sector giving rise to a pair of quaternionic structures. In the zero torsion case, the manifold is hyper Kähler. One may also consider the left-right asymmetric cases. The generalization to heterotic string theory is achieved by restricting the supersymmetry transformations to only one chiral sector and adding appropriate Chern-Simons terms to the torsion 3-form as in (3.2). In this case one also obtains a new condition, involving the gauge field, besides the ones listed in (4.1)-(4.4). These cases result in minor modifications of equations (4.1)-(4.4) and will be discussed in somewhat more detail in sections 3 and 5, where they are of immediate relevance to our study.

The above discussion implies that extended supersymmetry on the worldsheet imposes severe restrictions on the target space geometry and, therefore, on the allowed configurations of the background fields. On the other hand, it is well known that the string vacua, considered as conformal field theories, in general, have deformations under which the geometry of the target space changes. As discussed in the previous chapters, an important case is when the background fields have translational invariance along d of the space coordinates. The string vacua, then, admit deformations generated by the group $O(d,d;\mathrm{R})$, which, in heterotic string theory with abelian gauge fields is generalized to $O(d,d+16;\mathrm{R})$. This is a generalization of Narain's group to space-time dependent background fields [7, 8, 109, 110]. These deformations include the discrete duality transformations which are symmetries of the underlying conformal field theory [22, 18, 19, 111]. Now, consider a non-linear σ-model with extended supersymmetry on the worldsheet which also admits $O(d,d)$ (or $O(d,d+16)$) deformations. These deformations change the background fields in a highly non-trivial manner and, therefore, interfere with the constraints which the extended worldsheet supersymmetry imposes on the target space geometry. The issue of interest is to study the effect of the deformations of the background fields on the extended worldsheet supersymmetry.

The $N=1$ local supersymmetry on the worldsheet does not impose any restrictions on the background fields and is, therefore, trivially preserved under the above mentioned deformations. The extended supersymmetries, however, require the existence of complex structures on the target manifold and restrict its geometry. If the theory also has some isometries, then it admits deformations which change the target space geometry and field configuration, while, preserving the worldsheet conformal invariance. It has been known for sometime that theories with extended $N=2$ supersymmetry which can be represented in terms of chiral and twisted chiral superfields and which also have an isometry, admit a discrete duality transformation [100]. This duality transformation converts a chiral superfield into a twisted chiral one (or *vice versa*) and also changes the target space geometry. It has been shown that this duality is the same as the usual duality transformation which is a discrete subgroup of the $O(d,d)$ group, when applied to $N=2$ theories [18, 112, 113]. Recently, in [114] duality with respect to one isometry direction was considered in the more general case, when a superfield representation is not necessarily known, and the dual complex structures were obtained. It was further argued in [112, 113] that, starting from a theory with manifest $N=2$ extended supersymmetry, at least a subgroup of $O(d,d)$ deformations should preserve the supersymmetry. The outcome of a general $O(d,d)$ or $O(d,d+16)$ deformation of an arbitrary theory with extended supersymmetry is, however, far from clear. In this chapter, we consider the most general case and study the effect of deformations (including duality transformations) of the superstring and heterotic string backgrounds on the constraints which are imposed by extended worldsheet supersymmetries on the target manifold. Obtaining the transformation properties of the complex structures, it will be shown that the deformations preserve extended supersymmetries on the worldsheet and, hence, correspond to marginal deformations of the underlying superconformal field theory. They, however, do not preserve the Kähler structure [98, 99] and the product structure [100, 102, 103] on the target manifold. The explicit form of the transformations we have used are accurate to one-loop level in the σ-model perturbation theory, though, corrections are known to exist to all orders as discussed in the previous chapter. The calculation is simplified by introducing some quantities which transform nicely under $O(d,d)$ and $O(d,d+16)$ deformations.

This chapter is organized as follows. In section 2, we write the action of the $O(d,d)$ transformations in a linear form using the target space vielbeins. We, then, construct two connections induced by these transformations which are related by the worldsheet parity transformation. In section 3, we consider non-linear σ models with extended $N=2$ and $N=4$ supersymmetries and translational invariance along d of the target space coordinates. We find the transformations of the complex structures under $O(d,d)$ deformations and, using the connections introduced in section 2, show that the deformations preserve the extended worldsheet supersymmetry. We also comment on the form of perturbative corrections to $O(d,d)$ transformations in the context of $(4,4)$ theories. In section 4, we consider, in more detail, the transformations of the complex structures in relation to manifest superfield representation of theories with extended $N = 2$ supersymmetry. It is shown that, generically, the deformed theories do not have a manifest superfield representation. The case of discrete duality transformation with respect to d isometries is discussed with reference to its relation to a duality known in the $N = 2$ theories. In section 5, we linearize the action of the $O(d, d + 16)$ group on the heterotic string backgrounds and, using the induced connection, show the invariance of the extended $(0,2)$ and $(0,4)$ supersymmetries under the deformations. The results are summarized in section 6.

4.2 LINEARIZATION OF THE O(d, d) TRANSFORMATIONS AND THE INDUCED CONNECTIONS

In this section we will rewrite the action of the $O(d,d)$ group which deforms string vacua with d isometries, in terms of the target space vielbeins. We will consider background field configurations which are not necessarily block diagonal in form as in (1.2). We then discuss the action of the worldsheet parity transformation on the deformations and introduce two $O(d,d)$ induced connections. These connections will be useful in our analysis of extended worldsheet supersymmetry in section 3.

First of all, we describe the convention we have adopted in labeling the coordinates in the remaining part of this chapter: The D-dimensional space-time indices are written as $K, L, M, \ldots$ and the corresponding tangent space indices are written as $A, B, C, \ldots$. All fields are assumed to be independent of d of the space coordinates with indices denoted by $l, m, n, \ldots$. The remaining $D - d$ coordinates, on which the background fields can depend, include the time direction and carry indices $\mu, \nu, \gamma, \ldots$. We will, however, use the matrix notation and suppress the indices whenever possible. In that case, the first d rows and columns of a matrix are labeled by indices $l, m, n, \ldots$ and the remaining ones are labeled by the indices $\mu, \nu, \gamma, \ldots$. Having fixed our conventions, we start with a description of the $O(d,d)$ transformations in cases when the background fields G and B are not block diagonal in form as given in (1.2). This corresponds to a restriction of the $O(d, d+16)$ transformations discussed in the previous chapter (3.27) to the case with zero background gauge fields. The object which transforms nicely under the $O(d,d)$ transformations is now the $2D$ dimensional matrix, M, constructed out of the full background fields G_{MN} and B_{MN} as

$$M = \begin{pmatrix} G^{-1} & 1_D - G^{-1}B \\ 1_D + BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{4.6}$$

Note that, as far as the transformation properties of the background fields are concerned, M is arbitrary upto the addition of an $O(d,d)$ invariant matrix. To obtain an action on M, the group $O(d,d)$ is considered as a subgroup of the larger group $O(D,D)$ in its fundamental representation (the explicit embedding of $O(d,d)$ in $O(D,D)$ is given below). If Ω is an element of the $O(d,d)$ group constructed in this way, then the matrix M transforms under the adjoint action of Ω,

$$\widetilde{M} = \Omega M \Omega^T, \qquad \Omega \in O(d,d) \subset O(D,D) \tag{4.7}$$

At the one-loop level in the σ-model perturbation theory, this is also supplemented by an appropriate transformation of the dilaton field Φ which is not relevant for our purposes. The representation of $O(D,D)$ is chosen such that the equation defining the group elements Ω takes the form

$$\Omega^T L \, \Omega = L \,, \quad L = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix} \tag{4.8}$$

The explicit embedding of Ω in $O(D,D)$ is given by the following parametrization

$$\Omega = \begin{pmatrix} A_1 & C_1 \\ C_2 & A_2 \end{pmatrix},$$

where, A_i and C_i are $D \times D$ matrices defined as

$$A_i = \begin{pmatrix} \mathcal{A}_i & 0 \\ 0 & \eta_{D-d} \end{pmatrix}, C_i = \begin{pmatrix} \mathcal{C}_i & 0 \\ 0 & 0_{D-d} \end{pmatrix}; \quad \text{with} \quad \begin{pmatrix} \mathcal{A}_1 & \mathcal{C}_1 \\ \mathcal{C}_2 & \mathcal{A}_2 \end{pmatrix} \in O(d,d) \tag{4.9}$$

As discussed in Chapter 1, general $O(d,d)$ transformation can be parametrized in terms of its action on the background fields. The group elements given by $\mathcal{A}_2 = (\mathcal{A}_1^T)^{-1}, \mathcal{C}_1 = \mathcal{C}_2 = 0$, correspond to $GL(d,R)$ transformations and the ones given by $\mathcal{A}_2 = \mathcal{A}_1 = 1, \mathcal{C}_1 = 0, \mathcal{C}_2 = -\mathcal{C}_2^T$, correspond to constant gauge transformations of the antisymmetric tensor field, B_{MN}. The non-trivial deformations of the backgrounds are generated by elements from $O(d)\times O(d)/O(d)$. The $O(d)\times O(d)$ subgroup is parametrized as

$$\Omega=\frac{1}{2}\begin{pmatrix} S+R & S-R \\ S-R & S+R \end{pmatrix}, \text{ where, } S(R)=\begin{pmatrix} \mathcal{S}(\mathcal{R}) & 0 \\ 0 & \eta_{D-d} \end{pmatrix}, \ \mathcal{S},\mathcal{R}\in O(d). \tag{4.10}$$

The diagonal $O(d)$ subgroup, given by $\mathcal{S}=\mathcal{R}$, generates ordinary rotations and is already included in $GL(d,R)$. To get non-trivial deformations, one considers (4.10) modulo this subgroup. The usual duality transformations [18]-[22] with respect to isometries along coordinates X^i, correspond to a discrete subgroup of $O(d,d)$ generated by

$$\mathcal{S}=1\,,\ \mathcal{R}=1-2\varepsilon_i\,,\quad \text{where, } (\varepsilon_i)_{jk}=\delta_{ij}\delta_{ik} \tag{4.11}$$

The full generalized duality group, $O(d,d;\mathrm{Z})$, is generated by (4.11), $GL(d,\mathrm{Z})$ and discrete gauge transformations of B_{MN} [19]. In the following, however, we will only refer to (4.11) as duality transformations.

To simplify the manipulations, in the following, we rewrite the above transformations in terms of the metric vielbein. Using the inverse vielbein, e, and a matrix K defined as

$$G^{-1}=e\,\eta\,e^T,\qquad K=G+B, \tag{4.12}$$

we construct a $2D\times D$ rectangular matrix ξ as

$$\xi=\begin{pmatrix} e \\ Ke \end{pmatrix} \tag{4.13}$$

This is related to M by

$$M=\xi\,\eta\,\xi^T=\begin{pmatrix} e \\ Ke \end{pmatrix}\eta\begin{pmatrix} e^T & e^TK^T \end{pmatrix} \tag{4.14}$$

and transforms, under $O(d,d)$, as

$$\widetilde{\xi}=\Omega\,\xi\,,\qquad \Omega\in O(d,d)\subset O(D,D) \tag{4.15}$$

Note that the Lorentz indices are not affected by the transformations[1]. In the remaining part of this section, we will describe some constructions which are useful for our analysis of extended worldsheet supersymmetry in section 3.

[1]When $S=R$, one may perform compensatory transformations on the Lorentz indices so that the flat space vielbein is invariant under rotations.

Since the extended worldsheet supersymmetries are manifestly invariant under $GL(d, R)$ and the B_{MN} gauge transformations, in the following, we will mainly concentrate on the non-trivial deformations generated by $O(d) \times O(d)$ which are parametrized by the matrices S and R defined in (4.10). The transformations of the the vielbein e, the metric G, and the matrix $K = G + B$ under the action of this subgroup can easily be written in terms of a matrix $Q_-(S, R)$ as

$$\begin{aligned}\widetilde{e} &= Q_-(S, R)e \\ \widetilde{G}^{-1} &= Q_-(S, R)\, G^{-1} Q_-^T(S, R) \\ \widetilde{K} &= Q_-(S, -R) Q_-^{-1}(S, R)\end{aligned} \tag{4.16}$$

where,

$$\begin{aligned}Q_-(S, R) &= \frac{1}{2}\Big(S + R + (S - R)\, K \Big) \\ Q_-^{-1}(S, R) &= \frac{1}{2}\Big(S^T + R^T + (S^T - R^T)\, \widetilde{K}\Big)\end{aligned} \tag{4.17}$$

The second equation above follows from the fact that $\Omega^{-1}(S, R) = \Omega(S^T, R^T)$ and the subscript for Q has been chosen in anticipation of the transformation properties of the complex structures to be discussed in the next section. Note that though $O(d, d)$ is a global transformation, the matrix Q_- which implements the transformation on the vielbein becomes local through its dependence on $K(X) = G(X) + B(X)$. Also, since the conditions for the existence of extended supersymmetry on the worldsheet involve space-time derivatives of the background fields, it is desirable to find convenient expressions for the transformation of such quantities. To this end, first note that under the transformation (4.15), the quantity $\xi^T L\, \partial_\mu \xi$ is invariant. Now, consider a vector field $V(X)$ which under $O(d, d)$ transforms as $\widetilde{V} = Q_- V$. Corresponding to this transformation, we can construct an $O(d, d)$ induced connection as $e\, \partial_\mu e^{-1}$. However, it is more useful to introduce, using the above facts along with equation (4.16), a non-minimal $O(d, d)$ induced connection, ω_μ^-, given by

$$\omega_\mu^- = \frac{1}{2} e\eta\xi^T L\partial_\mu \xi e^{-1} - e\partial_\mu e^{-1}, \tag{4.18}$$

in terms of which,

$$\partial_\mu K = 2\, G\, \omega_\mu^- \tag{4.19}$$

Under an $O(d,d)$ deformation, this connection transforms to

$$\widetilde{\omega}_\mu^- = Q_- \, \omega_\mu^- Q_-^{-1} - \partial_\mu Q_- Q_-^{-1} \tag{4.20}$$

It should be emphasized that ω_μ^- is a connection in the space-time sense but is not the same as the natural torsion-full connections on a σ-model manifold (see equation (4.30) below).

On the worldsheet, the extended supersymmetries in the left moving sector and the right moving sector are interchanged by the worldsheet parity transformation, $\sigma \to -\sigma$. Under this transformation, $B \to -B$ and S and R are interchanged. The matrix Q_-, therefore, goes over to a new matrix Q_+, given by

$$\begin{aligned} Q_+ &= \frac{1}{2}\Big(S + R - (S - R)\, K^T \Big) \\ Q_+^{-1} &= \frac{1}{2}\Big(S^T + R^T - (S^T - R^T)\, \widetilde{K}^T \Big) \end{aligned} \tag{4.21}$$

Since G is invariant under the worldsheet parity, one expects $\widetilde{G}$ also to be invariant. In fact, using (4.17),(4.21) and (4.10), it can be shown that

$$\widetilde{G}^{-1} = Q_- \, G^{-1} \, Q_-^T = Q_+ \, G^{-1} \, Q_+^T \tag{4.22}$$

Now, we introduce a second connection corresponding to transformations of the form $\widetilde{V} = Q_+ V$. This connection, the relevance of which will become clear in the next section, is given by

$$\omega_\mu^+ = G^{-1} \, \omega^{-T}_{\ \mu} \, G, \tag{4.23}$$

and, under an $O(d,d)$ deformation, transforms as

$$\widetilde{\omega}_\mu^+ = Q_+ \, \omega_\mu^+ Q_+^{-1} - \partial_\mu Q_+ Q_+^{-1} \tag{4.24}$$

Having introduced two connections, we can also define the corresponding $O(d,d)$ covariant derivatives,

$$\mathcal{D}_\mu^\pm = \partial_\mu + \omega_\mu^\pm . \tag{4.25}$$

The actual form of the covariant derivative, obviously, depends on the particular representation chosen. At the end, note that the transformations of the two connections, (4.20) and (4.24), can also be written as

$$\widetilde{\omega}_\mu^\pm = Q_\mp \omega_\mu^\pm Q_\pm^{-1} . \tag{4.26}$$

4.3 EXTENDED SUPERSYMMETRY ON THE WORLDSHEET AND THE O(d, d) DEFORMATIONS

In this section, it will be shown that the deformations of the background fields generated by the $O(d,d)$ group preserve the conditions of extended $(2,2)$ and $(4,4)$ supersymmetries on the worldsheet. We will first consider the $(2,2)$ and then the $(4,4)$ extended supersymmetries. The result for the left-right asymmetric situations follows trivially. We will also comment on the possible restrictions on the form of perturbative corrections to the $O(d,d)$ transformation equations in the context of $(4,4)$ supersymmetry.

In superstring theory, the local $(1,1)$ supersymmetry can be promoted to an extended $(2,2)$ supersymmetry if background fields satisfy the conditions (4.1)-(4.4). Henceforth, we will only consider situations where the complex structures are independent of d of the coordinates X^m. To investigate the invariance of the these conditions under $O(d,d)$ deformations, we need the transformation properties of the complex structures $J^{\pm}$ under the deformations. These can be obtained by using the transformation of the metric as given in (4.22) and demanding that the deformations preserve equation (4.3). At this level, however, there is an ambiguity in determining the transformations of the complex structures because of the two possible ways in which a transformation of the metric can be written in (4.22). One way of resolving the ambiguity, obviously, is to find out which form of the transformations, if any, keep the conditions (4.4) invariant. But, a simpler and more illuminating method is to note that for infinitesimal background fields, $G = 1 + h, B = b$, equation (4.16) reduces to $\tilde{h} + \tilde{b} = R(h + b)S^T$. The quantity $h + b$ can also be interpreted as the polarization tensor in a vertex operator for the emission of massless states in superstring theory formulated in flat backgrounds If we consider correlation functions of this vertex operator which carry zero momentum along d of the space coordinates, then, the above transformation is equivalent to rotating, by different amounts, the left-moving and right-moving parts of the d bosonic coordinates along with that of their fermionic partners: $(\,\partial X, \psi_+\,) \to (\,R\partial X, R\psi_+\,)$ and $(\,\bar{\partial} X, \psi_-\,) \to (\,S\bar{\partial} X, S\psi_-\,)$. In order to keep the second supersymmetry intact, this implies that, to zeroth order in the backgrounds, the complex structures should transform as $\tilde{J}^+ = RJ^+R^T$ and $\tilde{J}^- = SJ^-S^T$. Also in flat backgrounds, $Q_- = S$ and $Q_+ = R$. Combining this fact with (4.22), and requiring the invariance of the hermiticity condition (4.3), the transformations of the complex structures under $O(d,d)$ are uniquely determined as

$$\begin{aligned}\tilde{J}^- &= Q_- J^- Q_-^{-1} \\ \tilde{J}^+ &= Q_+ J^+ Q_+^{-1}\end{aligned} \tag{4.27}$$

These transformations trivially preserve the conditions for the existence of almost complex structures (4.1). In the following, we analyze the covariance of conditions (4.2) and (4.4).

To simplify the analysis, we introduce two rank 3 $O(d,d)$ tensors, $\mathcal{J}^{\pm\lambda}$, defined as

$$\mathcal{J}^{\pm\lambda LN}_{K} = G^{LN} J^{\pm\lambda}{}_{K} - J^{\pm L}{}_{K} G^{N\lambda} - \delta^{\lambda}{}_{K}(J^{\pm}G^{-1})^{NL} + \delta^{L}{}_{K}(J^{\pm}G^{-1})^{N\lambda}, \tag{4.28}$$

Under an $O(d,d)$, they transform to

$$\tilde{\mathcal{J}}^{\pm\lambda L'N'}_{K'} = Q_{\pm}{}^{L'}_{L}\, Q_{\pm}{}^{N'}_{N}\, Q_{\pm}^{-1}{}^{K}{}_{K'}\, \mathcal{J}^{\pm\lambda LN}_{K}, \tag{4.29}$$

as can be seen using (4.16) and (4.27). Also note that, in terms of the induced connections $\omega^{\pm}_{\mu}$ (4.18)(4.23), the generalized connections (4.5) take the form

$$\Omega^{\pm N}_{ML} = \omega^{\pm}_{M}{}^{N}{}_{L} + (G\,\omega^{\pm}_{L}G^{-1})^{N}_{M} - G^{N\mu}(G\,\omega^{\pm}_{\mu})_{ML} \tag{4.30}$$

Using the above relations and the $O(d,d)$ tensors (4.28), the conditions of covariant constancy of the complex structures (4.4) can be rewritten as the following two equations (for $M = m, \mu$),

$$\begin{aligned}(\nabla^{\pm}_{m} J^{\pm})^{N}_{K} &= (G\,\omega^{\pm}_{\lambda})_{mL}\, \mathcal{J}^{\pm\lambda LN}_{K} = 0 \\ (\nabla^{\pm}_{\mu} J^{\pm})^{N}_{K} &= (\mathcal{D}^{\pm}_{\mu} J^{\pm})^{N}_{K} + (G\,\omega^{\pm}_{\lambda})_{\mu L}\, \mathcal{J}^{\pm\lambda LN}_{K} = 0\end{aligned} \tag{4.31}$$

where,

$$\mathcal{D}^{\pm}_{\mu} J^{\pm} = \partial_{\mu} J^{\pm} + [\,\omega^{\pm}_{\mu}\,,\, J^{\pm}\,]$$

are the two $O(d,d)$ induced covariant derivatives introduced in the previous section (4.25). The analysis of the invariance of these conditions under an $O(d,d)$ deformation is now straightforward. Using the transformation laws of the connections $\omega^{\pm}_{\mu}$ (4.20)(4.24) and tensors $\mathcal{J}^{\pm\lambda}$ (4.29), along with the fact that $(Q_{\pm})^{\mu}{}_{m} = (Q_{\pm}^{-1})^{\mu}{}_{m} = 0$, we obtain

$$\begin{aligned}(\tilde{\nabla}^{\pm}_{m'} \tilde{J}^{\pm})^{N'}_{K'} &= Q_{\mp}^{-1}{}^{m}{}_{m'}\, Q_{\pm}{}^{N'}_{N}\, Q_{\pm}^{-1}{}^{K}{}_{K'} (\nabla^{\pm}_{m} J^{\pm})^{N}_{K} = 0 \\ (\tilde{\nabla}^{\pm}_{\mu} \tilde{J}^{\pm})^{N'}_{K'} &= Q_{\pm}{}^{N'}_{N}\, Q_{\pm}^{-1}{}^{K}{}_{K'} \left[(\nabla^{\pm}_{\mu} J^{\pm})^{N}_{K} + Q_{\mp}^{-1}{}^{m}{}_{\mu} (\nabla^{\pm}_{m} J^{\pm})^{N}_{K} \right] = 0\end{aligned} \tag{4.32}$$

where, the vanishing of the right-hand sides follows from (4.31). This proves that the transformed complex structures are still covariantly constant with respect to the deformed generalized connections. The last equations to check are the integrability conditions (4.2) of the almost complex structures $J^{\pm}$. After some manipulations, the $O(d,d)$ deformed Nijenhuis tensors can be written as

$$\widetilde{N}^{\pm K'}_{\ \ M'N'} = Q_{\pm}^{-1M}{}_{M'}\, Q_{\pm}^{-1N}{}_{N'}\, Q_{\pm}{}^{K'}{}_{K} \left(N^{\pm K}{}_{MN} + \mathcal{J}^{\pm\lambda LP}{}_{M}\, G_{PN} \left[Q_{\pm}^{-1} \partial_\lambda Q_{\pm}\, ,\, J \right]^{K}{}_{L} \right) \tag{4.33}$$

which, on further manipulation, reduce to

$$\begin{aligned} \widetilde{N}^{\pm K'}_{\ \ M'N'} &= Q_{\pm}^{-1M}{}_{M'}\, Q_{\pm}^{-1N}{}_{N'}\, Q_{\pm}{}^{K'}{}_{K} \Big(N^{\pm K}{}_{MN} \\ &\quad \mp \left(\delta^K_L (GJ^{\pm})_{PN} - J^{\pm K}{}_L G_{PN} \right) \left(Q_{\pm}^{-1} (S-R) \right)^{Lm} (\nabla^{\pm}_m J^{\pm})^P{}_M \Big) = 0 \end{aligned} \tag{4.34}$$

Here, the vanishing of the deformed Nijenhuis tensors follows from equations (4.2) and (4.4). Thus, the deformed almost complex structures are integrable. Note that the integrability condition is preserved only for complex structures which are covariantly constant with respect to the torsionful connections. This completes the proof of the invariance of the $(2,2)$ extended supersymmetry on the worldsheet under $O(d,d)$ deformations.

The other possible extension of the $N=1$ supersymmetry is to $N=4$ [100, 102, 103]. In the $(4,4)$ case, this extension requires the existence, in each chiral sector separately, of three complex structures, $J_a; a=1,2,3$. Each one of the J_a's satisfies conditions (4.1)-(4.4) independently. In addition, the J_a satisfy an $SU(2)$ algebra, giving rise to a quaternionic structure on the target manifold,

$$J_a^{\pm} J_b^{\pm} = -\delta_{ab} + \epsilon_{abc} J_c^{\pm}.$$

Also, condition (4.2) is now generalized to the vanishing of the mixed Nijenhuis tensors,

$$N^{\pm K}_{(ab)MN} = J^{\pm L}_{(a\,M}\, \partial_{[L}\, J^{\pm K}_{b)\,N]} - J^{\pm L}_{(a\,N}\, \partial_{[L}\, J^{\pm K}_{b)\,M]} = 0$$

The invariance of the constraints of extended $(4,4)$ supersymmetry under a deformation follows from the above discussion for the $(2,2)$ case coupled with the fact that the transformations of the complex structures do not affect the $SU(2)$ index a. This proves the invariance of the extended $(4,4)$ supersymmetry under $O(d,d)$ deformations. Since supersymmetry is preserved in the left and right chiral sectors independently, the above analysis can be trivially generalized to any model with extended (p, q), $p, q = 0, 1, 2, 4$ supersymmetry.

The invariance of the $(4, 4)$ supersymmetry under the deformations may have an implication for the possible form of higher σ-model loop corrections to the $O(d, d)$ transformations. The form of the transformation given in (4.7), with M defined as in (4.6), is correct to one-loop in the σ-model perturbation theory. However, the arguments in [16, 23] and the analysis of [19] and [24] imply the existence of corrections, to all orders, to the transformation. On the other hand, if the non-renormalization theorems for the extended $(4, 4)$ supersymmetry on the worldsheet [102, 103] are valid, then, for these theories, the corrections to the $O(d, d)$ transformations must vanish. This restricts the possible form of the higher loop corrections in terms of the constraints which the $(4, 4)$ supersymmetry imposes on the background fields.

4.4 COMPLEX STRUCTURES, DUALITY AND MANIFEST N = 2 SUPERSYMMETRY

In this section we analyze the transformations of the complex structures and show that, starting from a theory with manifest $N = 2$ supersymmetry, the deformed theories do not, in general, admit a manifestly supersymmetric description in terms of chiral and twisted chiral superfields. We write the action of a general discrete duality transformation on the complex structures and discuss the case of Kähler manifolds.

The existence of extended $N = 2$ supersymmetry on the worldsheet severely constrains the target space geometry by requiring the existence of two complex structures $J^{\pm}$ satisfying equations (4.1)-(4.4). A simpler situation arises when the torsion, H_{MNP}, on the target space is zero. Conditions (4.1)-(4.4), then, do not distinguish between the two complex structures which can, therefore, be chosen to be the same. This complex structure is covariantly constant with respect to the Riemmanian connection and the corresponding manifold is Kähler. A non-linear σ-model defined on a Kähler manifold can be written, in a manifestly $(2, 2)$ supersymmetric form, in terms of the Kähler potential as a function of $N = 2$ chiral superfields. In the presence of torsion, a superfield formulation of the theory is not always known. However, there is a special class of theories with $H_{MNP} \neq 0$ which still admit a manifestly supersymmetric description in terms of chiral and twisted chiral superfields.

These are the theories in which the two complex structures commute and are, therefore, simultaneously diagonalizable [100, 102, 103]. We can always choose a canonical basis in which J^- takes the form $J^{-a}{}_b = i\delta^a{}_b$, $J^{-\bar{a}}{}_{\bar{b}} = -i\delta^{\bar{a}}{}_{\bar{b}}$, where, $a, \bar{a} = 1, \ldots, D/2$ label the holomorphic and antiholomorphic coordinates. In this basis, J^+ is also diagonal but the arrangement of $\pm i$'s on the diagonal depend on the number of twisted chiral superfields or, equivalently, the form of B_{MN}. Such a manifold is said to have a product structure.

The Kähler and product structures, however, are very special and are not, in general, preserved under non-trivial $O(d,d)$ deformations of the complex structures. To investigate this point, we first consider the case of flat backgrounds. In this case, as discussed above equation (4.27), the transformations can be interpreted as independent d-dimensional rotations of the left and right moving parts of the bosonic and fermionic coordinates inside correlation functions which carry zero momentum along d of the space directions. Modulo ordinary d-dimensional rotations and parity transformations, which are symmetries of the theory, we can choose $\mathcal{S} = 1$ in (4.27). The complex structures then transform as $J^- \to J^-$, $J^+ \to RJ^+R^T$. Note that R is rotation involving the real coordinates. Therefore, if we choose the usual real representation for R, then, J^+ also has to be written in the real coordinate basis. If we write J^+ in the diagonal basis (which is not real), then R also has to be transformed appropriately. It can be explicitly checked that starting from commuting complex structures, non-trivial R transformations that preserve this commuting nature are (i) the ones which flip the $+i$ and $-i$ eigenvalues of J^+ written in the canonical basis of J^-, (ii) the ones that correspond to $O(2)$ rotations involving the real and imaginary parts of the same complex coordinate (note that a rotation which mixes the real and imaginary parts of different complex coordinates does not qualify). The first case corresponds to converting a chiral superfield into a twisted chiral one, or *vice versa*. It can be easily checked that these transformations form the discrete duality subgroup (4.11) of $O(d,d,R)$. This shows the connection between the usual duality and the $N=2$ duality of [100]. Case (ii) corresponds to $(O(2)\times O(2))/O(2)$ deformations. The remaining elements of $(O(d)\times O(d))/O(d)$ do not, in general, preserve the commuting nature of the complex structures even when the effects of background fields are ignored. Therefore, in the following, where the background fields are not ignored, we will concentrate only on cases (i) and (ii) above.

In the presence of non-trivial background fields, it can be explicitly checked that $(O(2)\times O(2))/O(2)$ deformations do not generically preserve the commuting nature of the complex structures. One exception is when the manifold is Kähler and the metric is independent of one of the complex coordinates, say z_1, and its conjugate. then it can be explicitly checked that an $O(2,2)$ deformation involving only the real and imaginary parts of z_1 does not change the complex structures. The manifold, thus, remains Kähler even after the deformation. We, now, turn to the case of discrete duality transformations (4.11). A duality with respect to d of the coordinates, on which the background fields do not depend, is generated by $\mathcal{S}=1\,,\mathcal{R}=1-2\,\sum_{i=1}^{d}\varepsilon_i$. The dual complex structures, obtained from (4.27), are given by

$$\widetilde{J}^{-}=\begin{pmatrix} (KJ^{-})^m{}_l(\mathcal{K}^{-1})^l{}_n & -(KJ^{-})^m{}_l(\mathcal{K}^{-1})^l{}_pK^p{}_\nu+(KJ^{-})^m{}_\nu \\ J^{-\mu}{}_l(\mathcal{K}^{-1})^l{}_n & -J^{-\mu}{}_l(\mathcal{K}^{-1})^l{}_pK^p{}_\nu+J^{-\mu}{}_\nu \end{pmatrix} \tag{4.35}$$

and

$$\widetilde{J}^{+}=\begin{pmatrix} (K^TJ^{+})^m{}_l(\mathcal{K}^{-1T})^l{}_n & (K^TJ^{+})^m{}_l(\mathcal{K}^{-1T})^l{}_pK^{Tp}{}_\nu-(K^TJ^{+})^m{}_\nu \\ -J^{+\mu}{}_l(\mathcal{K}^{-1T})^l{}_n & -J^{+\mu}{}_l(\mathcal{K}^{-1T})^l{}_pK^{Tp}{}_\nu+J^{+\mu}{}_\nu \end{pmatrix} \tag{4.36}$$

where, $\mathcal{K}_{mn}=K_{mn}\,;m,n=1,\ldots,d$ is a $d\times d$ submatrix of K, and the indices of K are raised and lowered by a flat metric which appears in the $O(d,d)$ transformations and has not been explicitly written here. For $d=1$, these expressions were obtained in [114]. From the above, one can see that even if the complex structures, J^+ and J^-, commute, their duals in general do not commute except for some restricted backgrounds. A situation which can be discussed in some generality is the Kähler manifold with isometries, on which, the two complex structures are chosen to be equal. If the isometries are along some components of the canonical coordinates, then, one can explicitly check that the dual complex structures are still commuting. This reproduces the duality of $N=2$ theories discovered in [100] when the starting theory is defined on a Kähler manifold. However, for more general isometries, after a duality transformation with respect to d isometries, the dual complex structures do not commute, even for this restricted class. The non-zero terms in the commutator are, however, proportional to the components of the Kähler form, GJ, in the isometry directions. Since the Kähler form is antisymmetric, for $d=1$ the dual complex structures commute without further restrictions on the background fields.

From the above discussion it follows that though the deformations preserve the extended $N = 2$ supersymmetry, the Kähler and product structures are not, in general, preserved. In such cases, the deformed theories do not have a formulation in terms of chiral and twisted chiral superfields. This implies that, starting from a conformally invariant non-linear σ-model with manifest extended $N = 2$ supersymmetry and some isometries, one can construct a large class of new theories with $N = 2$ extended supersymmetry, for which, a superfield representation is not known. This is achieved without solving constraints (4.1)-(4.4) and the β-function equations. Duality transformations applied to the models of [100] preserve the superfield representation. The discussion also clarifies the connection between the duality transformations (4.11) and the duality of $N = 2$ theories discussed in [100].

4.5 THE O(d, d + 16) TRANSFORMATIONS AND EXTENDED WORLDSHEET SUPERSYMMETRY IN THE HETEROTIC STRING THEORY

In this section we rewrite the action of the $O(d, d+16)$ group on the heterotic string theory backgrounds, including the gauge Chern-Simons term, in terms of the metric vielbein. We then construct a connection induced by these transformations and show that the extended worldsheet supersymmetry in heterotic string theory is preserved under the deformations.

As discussed in Chapter 3, in heterotic string theory, the $O(d, d)$ group of deformations of the string vacua with d isometries is enlarged to an $O(d, d + p)$ group provided all the background gauge fields belong to a subgroup that commutes with p of the Cartan generators of the gauge group [7, 23, 60]. For simplicity, we consider the case when $p = 16$ and, therefore, all background gauge fields are abelian. As in the non-heterotic case, the action of the $O(d, d+16)$ group on the background fields is given in terms of a $(2D + 16) \times (2D + 16)$ dimensional matrix M constructed from G_{MN}, B_{MN} and the gauge fields A^I_M, where, $I = 1, \ldots, 16$ is the gauge group index [23]. The matrix M transforms under the adjoint action of the $O(d, d + 16)$ group embedded in the fundamental representation of $O(D, D + 16)$. As in section 2, we rewrite the transformations, in a slightly modified form, in terms a

$(2D+16)\times(D)$-dimensional matrix ξ defined as

$$\xi = \begin{pmatrix} e \\ K\,e \\ -A\,e \end{pmatrix} \tag{4.37}$$

where, A is a $16\times D$-dimensional matrix and K is given by

$$K_{MN} = G_{MN} + B_{MN} + \frac{1}{2}\,A^I_M\,A^I_N \tag{4.38}$$

The matrix M, mentioned above, can now be constructed in terms of ξ as[2]

$$M = \xi\,\eta\,\xi^T = \begin{pmatrix} e \\ Ke \\ -Ae \end{pmatrix} \eta \begin{pmatrix} e^T & e^T K^T & -e^T A^T \end{pmatrix} \tag{4.39}$$

Under an $O(d,d+16)$ transformation, ξ transforms as

$$\tilde{\xi} = \Omega\xi\,, \quad \Omega \in O(d,d+16) \subset O(D,D+16) \tag{4.40}$$

where, the representation is chosen such that the defining equation for Ω takes the form

$$\Omega^T\,L\,\Omega = L\,, \quad L = \begin{pmatrix} 0 & 1_D & 0 \\ 1_D & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix} \tag{4.41}$$

These transformations include the $GL(d,R)$ transformations and gauge transformations of B_{MN} and A^I_M. The non-trivial deformations are generated by the elements from $(O(d+16)\times O(d))/O(d)$. These are parametrized as

$$\Omega = \frac{1}{2}\begin{pmatrix} S+R & S-R & -R_1^T \\ S-R & S+R & R_1^T \\ -R_2 & R_2 & 2R_3 \end{pmatrix} \tag{4.42}$$

modulo the subgroup generated by $S=R$, where, S and R are defined as in equation (4.10), though the submatrix $\mathcal{R}$ of R is no longer necessarily an element of $O(d)$. Also, R_3 is a 16×16-dimensional matrix and $R_{1,2}$ are $16\times D$-dimensional matrices of the form

$$R_{1,2} = \begin{pmatrix} \sqrt{2}\,\mathcal{R}_{1,2} & 0_{16\times(D-d)} \end{pmatrix}\,, \quad \text{where,} \begin{pmatrix} \mathcal{R} & \mathcal{R}_1^T \\ \mathcal{R}_2 & R_3 \end{pmatrix} \in O(d+16)$$

[2]Note that the fields here are related to those of Chapter 3 by $B\to -B, K\to -K, A\to -A/\sqrt{2}$. This slight modification has been introduced to absorb some signs and factors in the definition of ξ.

Under the deformations generated by the above elements, the background fields transform as

$$\begin{aligned}
\tilde{e} &= Q\, e \\
\widetilde{K} &= \frac{1}{2}\Big((S-R) + (S+R)\,K - R_1^T\,A\Big)\,Q^{-1} \\
\tilde{A} &= \frac{1}{2}\Big(R_2 - R_2\,K + 2\,R_3\,A\Big)\,Q^{-1}
\end{aligned} \tag{4.43}$$

where, Q is given by

$$Q = \frac{1}{2}\Big((S+R) + (S-R)\,K + R_1^T\,A\Big) \tag{4.44}$$

The transformation of the metric can be obtained from that of the inverse vielbein e.

Now, as in section 2, we introduce an $O(d,d+16)$ induced connection as follows. Note that, under $O(d,d+16)$ deformations (4.40), the quantity $\xi^T\,L\,\partial_\mu\,\xi$ is invariant and, from (4.43), the quantity $e\,\partial_\mu\,e^{-1}$ transforms as a connection corresponding to transformations of the type $\tilde{V} = QV$. Using these two quantities, we construct a non-minimal $O(d,d+16)$ induced connection ω_μ as

$$\omega_\mu = \frac{1}{2}\,e\eta\xi^T L\partial_\mu\xi e^{-1} - e\partial_\mu e^{-1}, \tag{4.45}$$

which, in a more recognizable form, can be written as

$$\omega_\mu = \frac{1}{2}\,G^{-1}\,(\partial_\mu\,K - A^T\,\partial_\mu\,A) \tag{4.46}$$

Under an $O(d,d+16)$ deformation, ω_μ transforms to

$$\tilde{\omega}_\mu = Q\,\omega_\mu Q^{-1} - \partial_\mu Q\,Q^{-1} \tag{4.47}$$

Using the above construction, we set out to prove the invariance of the extended worldsheet supersymmetry in heterotic string theory under the $O(d,d+16)$ deformations.

In heterotic string theory, the local $(0,1)$ worldsheet supersymmetry can be extended to a $(0,2)$ supersymmetry in a way similar to the superstring case [101, 104, 102, 103]. The target manifold is required to admit an almost complex structure J with vanishing Nijenhuis tensor, $N^K_{MN} = 0$, and a hermitian metric, $J^T\,G\,J = G$. The difference with the superstring case, however, is that the torsion tensor H_{MNK} now also contains a gauge Chern-Simons term[3] coming from the one-loop chiral anomaly, $H_{MNK} = (1/2)(\partial_M B_{NK} + \cdots) +$

[3]In general, the torsion also contains the Lorentz Chern-Simons term which is a higher derivative term and will not be considered here.

$(1/4)(A^I_M F^I_{NK} + \cdots)$, where, the dots denote cyclic permutations and we have restricted ourselves to abelian gauge fields. This modifies the generalized connection and, thus, the condition of covariant constancy of the complex structure. With this modification in mind, the conditions for the existence of extended supersymmetry in heterotic string theory are again given by equations (4.1)-(4.4) for $J = J^-$. The only extra condition is the constraint on the gauge field background,

$$F^I_{KL}\, J^L{}_M - F^I_{ML}\, J^L{}_K = 0 \tag{4.48}$$

The transformation of the complex structure J under an $O(d, d+16)$ deformation is obtained by requiring the covariance of the hermiticity condition of the metric and is given by $\tilde{J} = Q\, J\, Q^{-1}$, with Q as given by (4.44). This also preserves $J^2 = -1$. To investigate the covariance of the remaining conditions, it is convenient to write them in terms of the induced connection ω_μ (4.46) and a rank 3 tensor, $\mathcal{J}^{\lambda LN}_K \equiv \mathcal{J}^-{}^{\lambda LN}_K$ (4.28). First, consider the covariant constancy condition $\nabla_M J = 0$. It turns out that, even in the presence of the gauge Chern-Simons term, the relation between the generalized connection Ω^- and the $O(d, d+16)$ induced connection (4.46) is given by equation (4.30). Therefore, the covariant constancy condition again takes the form given in (4.31). Next, we consider the condition on the gauge fields (4.48). It is easily seen that this equation can be rewritten as

$$\partial_\lambda\, A^I{}_L\, \mathcal{J}^{\lambda LN}_K\, G_{NM} = 0 \tag{4.49}$$

Using the above equations in terms of the $O(d, d+16)$ covariant variables, a straightforward calculation shows that, after a deformation, the tensors $N^K_{MN}, \nabla_M J$ and the left hand side of equation (4.49) transform into linear combinations of each other and, therefore, remain equal to zero. This proves the invariance of the $(0, 2)$ worldsheet extended supersymmetry in heterotic string theory under the $O(d, d+16)$ deformations of the background fields. The generalization to $(0, 4)$ supersymmetry follows from the discussion, in section 3, of extended $(4, 4)$ supersymmetry in superstring theory.

4.6 SUMMARY

We have shown that the $O(d, d; \mathrm{R})$ deformations of the superstring vacua and the $O(d, d+16; \mathrm{R})$ deformations of the heterotic string vacua can be rewritten, in a simpler way, in terms of the target space vielbeins. Though, these transformations are global in the space-time sense, their non-linear action on the background fields enables us to construct some induced space-time connections.

We then obtain the transformations of the complex structures, associated with extended supersymmetries on the worldsheet, under the above deformations. Using the induced connections and some tensors which transform covariantly under the deformations, we show that the $O(d,d;\mathrm{R})$ deformations of the superstring vacua and the $O(d,d+16;\mathrm{R})$ deformations of the heterotic string vacua preserve extended supersymmetries on the worldsheet. This is provided the complex structures have the same translational invariances as the massless background fields of the theory. The transformations, therefore, generate marginal deformations of the associated superconformal field theories.

In the case of extended $(2,2)$ supersymmetry, we discuss the transformations of the complex structures in relation to the superfield representation of the deformed theories. It is shown that generic deformations do not preserve the known superfield representations of the theories in terms of chiral and twisted chiral superfields. We also write down, explicitly, the transformations of the complex structures under the target space duality. The discussion clarifies the relation between the above duality transformation and a duality in the $N=2$ theories discussed by Gates, Hull and Roček [100]. We also comment on the possible form of the perturbative corrections to the $O(d,d)$ transformations in the context of $(4,4)$ theories.

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