$O(d,d)$ TRANSFORMATIONS, MARGINAL DEFORMATIONS AND THE COSET CONSTRUCTION IN WZNW MODELS

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ABSTRACT

We show the connection between $O(2,2)$ transformations and marginal deformations in the $SU(2)_k$ WZNW model and briefly discuss the generalization to other models. We also explain why the axial and vector cosets can be obtained by fixed $O(2,2)$ transformations of the $SU(2)$ theory.

1. Introduction

It is well known that the propagation of a string in a background of its massless excitations is described by a non-linear σ-model in two dimensions in which the background fields appear as couplings. In the bosonic string theory the massless background fields are the graviton field, G_{MN}, the anti-symmetric tensor field, B_{MN}, and the dilaton field, Φ. The superstring theory also contains the fermionic superpartners of these fields. The results we are going to discuss in the following are valid for the bosonic string as well as the superstring with the fermionic backgrounds set to zero. The conformal invariance of the σ-model at the quantum level requires that the β-functions corresponding to various couplings should vanish. This leads to a set of differential equations for the background fields which generalize the Einstein's equation in general relativity. The β-function equations can also be obtained as equations of motion from the low-energy effective action of string theory. A configuration of the background fields which solves these equations, defines a conformally invariant non-linear σ-model. It should be mentioned that the low-energy effective action and, therefore, the conformally invariant background fields receive corrections order by order in the σ-model perturbation theory.

It is known that when the background fields are restricted to configurations which are independent of d of the space-time directions, then the low-energy effective action, evaluated for this restricted class of backgrounds, is invariant under the action of an $O(d,d)$ group [1,2,3,4,5,6,7,8]. The action of this group on the background fields is given in terms of a $2D \times 2D$ matrix M which is constructed from the background fields G_{MN} and B_{MN} as follows

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{1}$$

Here, D is the total space-time dimension. The $O(d,d)$ group now acts on the background fields by transforming the matrix M and the dilaton field Φ as

$$M \to M' = \Omega M \Omega^T \tag{2}$$

$$\Phi \to \Phi' = \Phi + \ln\sqrt{\det G' / \det G} \tag{3}$$

Here, Ω is an $O(d,d)$ group element which is appropriately embedded in an $O(D,D)$ group as described below [5,7]. The transformations of G_{MN} and B_{MN} can be uniquely determined from Eq. (2). It turns out that these transformations contain the usual $GL(d)$ transformations of the background fields and some gauge transformations of the antisymmetric tensor field as subgroups. Since the transformations belonging to these two subgroups are symmetries of the full theory, they do not give rise to new background field configurations and will be refered to as trivial $O(d,d)$ transformations. The non-trivial transformations are parametrized as follows [3]: Let us write the matrices G and B in Eq. (1) in such a way that the first d columns and the first d rows are labeled by the d coordinates on which the background fields do not depend. The non-trivial transformations Ω are then parametrized as

$$\Omega = \frac{1}{2}\begin{pmatrix} \mathcal{R}+\mathcal{S} & \mathcal{R}-\mathcal{S} \\ \mathcal{R}-\mathcal{S} & \mathcal{R}+\mathcal{S} \end{pmatrix}, \text{ where}, \ \mathcal{R},\mathcal{S} = \begin{pmatrix} R,S & 0 \\ 0 & 1_{D-d} \end{pmatrix} \tag{4}$$

Here, R and S are element of the group $O(d)$ in the fundamental representation. There is, however, still a redundancy as for $R = S$ Eq. (2) corresponds to d-dimensional rotations already contained in $GL(d)$. The non-trivial $O(d,d)$ transformations are, therefore, parametrized by an $O(d) \otimes O(d)$ subgroup modulo the diagonal $O(d)$ subgroup. For generic background field configurations, the group elements Ω corresponding to this coset act non-linearly on the background fields and are not symmetries of the full unrestricted low-energy effective action. The transformations generated by these elements, therefore, relate background field configurations which are physically distinguishable and correspond to different conformal field theories. The $O(d,d)$ group also contains the target space duality transformations as a discrete subgroup [7]. These duality transformations are symmetries of the underlying conformal field theory thouth they change the target space geometry in a non-trivial way. To understand the connection between the $O(d,d)$ transformations of the background fields and the marginal deformations of the associated conformal field theories, we focus on theories for which it is possible to identify the marginal deformation operators easily. An important class of such theories are given by the Wess-Zumino-Novikove-Witten (WZNW) models. In the next section we consider the simplest such theory, the $SU(2)_k$ WZNW model. We show that the $O(2,2)$ transformations of this model correspond to marginal deformations of the theory. This gives us a method for explicitly constructing the full one-parameter family of marginal deformations of the $SU(2)_k$ model. For generalizations see the literature [10,11]. In section 3, we discuss the coset construction and give an explanation for the observation [16] that the axial and vector cosets of the $SU(2)$ theory can also be obtained by fixed $O(2,2)$ transformations of the original theory.

2. $O(d,d)$ Transformations and Marginal Deformations

In this section, we will explicitly consider the $O(2,2)$ transformations of the $SU(2)_k$ WZNW model omitting some details which has been given elsewhere [10]. At the end we will briefly discuss the extension to general sigma models with abelian chirally conserved currents [11].

We find it convenient to work with a parametrization of the $SU(2)$ group manifold in which the action for the corresponding WZNW model takes the form

$$S(\phi,\theta,\tilde{\theta}) = \frac{k}{2\pi}\int d^2z\Big(\frac{1}{4}\bar{\partial}\phi\partial\phi + \sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta + \cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\partial\tilde{\theta} + \cos^2\frac{\phi}{2}(\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta)\Big) \quad (5)$$

Here, $\theta, \tilde{\theta}$ and ϕ are coordinates on the group manifold. From this, we can read off the metric and the antisymmetric tensor field as

$$G = \begin{pmatrix} k/4 & 0 & 0 \\ 0 & k\sin^2\frac{\phi}{2} & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \\ 0 & -k\cos^2\frac{\phi}{2} & 0 \end{pmatrix} \quad (6)$$

It is well known that the action given in Eq. (5) defines a conformal field theory with an integrable marginal perturbation. This perturbation is generated by a $(1,1)$ operator of the form $J\bar{J}$, where, J and $\bar{J}$ are holomorphic and anti-holomorphic conserved currents generating the Cartan subalgebra of the $SU(2)$ algebra [9]. The perturbed theory is given by the action $S' = S + \delta\lambda\int d^z J\bar{J}$, where, $\delta\lambda$ is an infinitesimal parameter. With our parametrization of the $SU(2)$ group manifold, the chiral currents take the form

$$\begin{aligned} J &= k\,(\sin^2\tfrac{\phi}{2}\partial\theta + \cos^2\tfrac{\phi}{2}\partial\tilde{\theta}\,) \\ \bar{J} &= k\,(-\sin^2\tfrac{\phi}{2}\bar{\partial}\theta + \cos^2\tfrac{\phi}{2}\bar{\partial}\tilde{\theta}\,) \end{aligned} \quad (7)$$

The background fields G and B given in Eq. (6) are independent of two of the coordinates θ and $\tilde{\theta}$. This allows us to perform an $O(2,2)$ transformation on the $SU(2)_k$ WZNW model as described in section 1. The non-trivial transformations are given by Eq. (4) with R and S chosen as

$$R = S^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad (8)$$

The transformed metric, anti-symmetric tensor field and dilaton are now obtained from Eqs. (2) and (3). It turns out that the new metric has conical singularities at $\phi = 0$ and π which are removed by the scalings $\theta \to \theta/(\cos\alpha - k\sin\alpha)$ and $\tilde{\theta} \to \tilde{\theta}/\cos\alpha$. This restores the periodicities of θ and $\tilde{\theta}$ to 2π. We also make a further transformation $B_{\tilde{\theta}\theta} \to B_{\tilde{\theta}\theta} + L$, where L is a constant given by

$$L = \cos\alpha(k\cos\alpha + \sin\alpha) \quad (9)$$

Notice that the above scalings and the shift of the B field are all contained in the $O(2,2)$ group. After these transformations, the new σ-model action becomes

$$\begin{aligned} S_{(\alpha)}(\phi,\theta,\tilde{\theta}) &= \frac{k}{2\pi}\int d^2z\Big(\frac{1}{4}\bar{\partial}\phi\partial\phi + \frac{1}{\Delta}(\cos\alpha - k\sin\alpha)^2\sin^2\frac{\phi}{2}\bar{\partial}\theta\partial\theta \\ &+ \frac{1}{\Delta}\cos^2\alpha\cos^2\frac{\phi}{2}\bar{\partial}\tilde{\theta}\partial\tilde{\theta} - \frac{1}{\Delta}\cos^2\alpha\sin^2\frac{\phi}{2}(\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta)\Big) \end{aligned} \tag{10}$$

where, Δ is given by

$$\Delta = \cos^2\alpha + k\,(k\sin^2\alpha - 2\sin\alpha\cos\alpha)\cos^2\frac{\phi}{2} \tag{11}$$

The new dilaton field Φ is given by

$$\Phi = -\ln\Delta \tag{12}$$

Although the explicit form of the transformed action is valid only in the large k limit, it is known that corrections to it exist to all orders in $1/k$ [3,5,7].

The action (10) and the dilaton (12) describe a one-parameter family of of conformal field theories parametrized by α. To find the relation between the transformed and the untransformed models, note that the action $S_{(\alpha)}(\phi,\theta,\tilde{\theta})$ is invariant under the holomorphic and anti-holomorphic shifts of the coordinates given by

$$\delta\theta = \frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2}v(z)\,, \qquad \delta\tilde{\theta} = \frac{1}{2}\frac{1}{\cos^2\alpha}v(z) \tag{13}$$

$$\bar{\delta}\theta = -\frac{1}{2}\frac{1}{(\cos\alpha - k\sin\alpha)^2}\bar{v}(\bar{z})\,, \qquad \bar{\delta}\tilde{\theta} = \frac{1}{2}\frac{1}{\cos^2\alpha}\bar{v}(\bar{z}) \tag{14}$$

This leads to the existence of the chirally conserved currents of the form

$$J_{(\alpha)} = \tfrac{k}{\Delta}(\sin^2\tfrac{\phi}{2}\partial\theta + \cos^2\tfrac{\phi}{2}\partial\tilde{\theta}) = J/(\Delta) \tag{15}$$

$$\bar{J}_{(\alpha)} = \tfrac{k}{\Delta}(-\sin^2\tfrac{\phi}{2}\bar{\partial}\theta + \cos^2\tfrac{\phi}{2}\bar{\partial}\tilde{\theta}) = \bar{J}/(\Delta) \tag{16}$$

Now, we compare two nearby theories on the $O(2,2)$ deformation line parametrized by α. In terms of the currents $J_{(\alpha)}$ and $\bar{J}_{(\alpha)}$ the result can be written as

$$S_{(\alpha+\delta\alpha)} = S_{(\alpha)} + \frac{\delta\alpha}{\pi}\cos\alpha(\cos\alpha - k\sin\alpha)\int d^2z\, J_{(\alpha)}\bar{J}_{(\alpha)} \tag{17}$$

The above equation shows that appropriate $O(2,2)$ transformations of the $SU(2)_k$ WZNW model generate a continuous line of conformal field theories along which any two adjacent theories are related by marginal perturbations. As an exercise, the above analysis can also be performed for the $SU(2)\otimes SU(2)$ model [10]. The generalization to any non-linear σ-model with chiral isometries was considered by Henningson and Nappi [11]. In this work, it was shown that if the theory has d_L chiral and d_R anti-chiral conserved currents, then, all its marginal deformations of the type $\sum_{a,\bar{b}} C_{a\bar{b}} J^a \bar{J}^{\bar{b}}$ are generated by appropriately chosen $O(d_L + d_R, d_L + d_R)$ transformations. All the remaining $O(d_L + d_R, d_L + d_R)$ transformations have a trivial action on the theory.

3. Marginal Deformations and the Coset Construction

In this section we consider the gauging of the $O(2,2)$ deformed $SU(2)$ WZNW model. We then use the result to understand, in the context of our $SU(2)$ theory, an observation made by Kumar [16]. This observation relates the cosets based on a theory with chirally conserved currents to certain $O(d,d)$ transformations of the original theory.

It is well known that a conformal field theory based on a Kac-Moody algebra can also be written in terms of a WZNW Lagrangian defined on the corresponding group manifold. In this formulation, the GKO coset construction is implemented by coupling the chiral currents of the theory to non-dynamical gauge fields and then integrating out the gauge fields after fixing the gauge. This procedure can also be extended to general non-linear σ-models with chirally conserved currents. Due to the presence of the Wess-Zumino term, the introduction of gauge fields coupled to local isometry currents is not straightforward in these models. In particular, it cannot in general be achieved by a covariant derivative replacement [12,13]. However, for the type of models of interest to us, there exist a prescription for introducing the gauge fields by a covariant derivative replacement in such a way that conformal invariance is also preserved [10]. This is achieved by adding a total derivative term to the action, prior to gauging, such that the chiral currents with respect to which the coset is constructed become equal to the corresponding Noether currents.

Using the above mentioned prescription [10], we consider the gauging of the $O(2,2)$ deformed $SU(2)_k$ model given is Eq. (10). There are two gaugings possible: $i)$ the axial gauging, in which the translations of the coordinate $\widetilde{\theta}$ are made local and $ii)$ the vector gauging, in which the translations of the coordinate θ are made local. We first consider the axial gauging. Due to the judicious choice of the constant L in Eq. (9), the gauged action can be directly obtained by a covariant derivative replacement and is given by

$$S^{\text{gauged}}_{(\alpha)}(\phi,\theta,\widetilde{\theta},A) = S_{(\alpha)}(\phi,\theta,\widetilde{\theta}) - \frac{1}{2\pi}\int d^2z\left(A\bar{J}_{(\alpha)} + \bar{A}J_{(\alpha)} - \frac{k\cos^2\frac{\phi}{2}}{\Delta\cos^2\alpha}A\bar{A}\right) \qquad (18)$$

However, on fixing the gauge ($\widetilde{\theta}=0$) and integrating out the gauge field, one finds that the deformation parameter α disappears from the coset model. One, thus, ends up with the well-known $SU(2)/U(1)$ axial coset given by [14,15]

$$ds^2 = (k/4)d\phi^2 + k\tan^2(\phi/2)\,d\theta^2\,, \qquad e^{-\Phi} = \cos^2\frac{\phi}{2} \qquad (19)$$

The reason for the disappearance of α lies in the freedom to redefine the gauge field.

In fact, substituting

$$A = \cos\alpha\sqrt{\Delta}A' + \cos\alpha(\cos\alpha - \sqrt{\Delta})\frac{J}{k\cos^2\frac{\phi}{2}} \tag{20}$$

in Eq. (18), with a similar expression for $\bar{A}$, we find that the gauged action is independent of α :

$$S^{\rm gauged}_{(\alpha)}(\phi,\theta,\widetilde{\theta},A) = S^{\rm gauged}_{(\alpha=0)}(\phi,\theta,\widetilde{\theta},A') \tag{21}$$

The Jacobian following from the transformation (20) modifies the path integral measure and readjusts the dilaton field to zero.

The analysis of the vector gauging of the $O(2,2)$ deformed $SU(2)$ model is similar to the axial case. However, in this case, in order to obtain the gauged action by a covariant derivative substitution, the constant L of Eq. (9) has to be replaced by

$$L = \cos\alpha(k\cos\alpha + \sin\alpha) - k \tag{22}$$

This dose not alter the analysis of section 2. In this case, after gauge fixing ($\theta = 0$) and integrating out the gauge field, one obtains the vector $SU(2)/U(1)$ coset

$$ds^2 = (k/4)d\phi^2 + k\cot^2(\phi/2)\,d\widetilde{\theta}^2\,, \qquad e^{-\Phi} = \sin^2\frac{\phi}{2} \tag{23}$$

We are now in a position to explain the origin of the observation [16] that, in non-linear σ-models with chirally conserved currents [6,7], certain fixed $O(d,d)$ transformations produce a coset of the original theory plus free bosons. The problem is to understand why an $O(d,d)$ transformation should give rise to a coset of the original theory. We try to understand this in the context of our deformed $SU(2)$ theory. It is easy to check that for $\alpha = -\pi/2$ the deformed action given by Eq. (10) reduces to the axial coset (19) plus a compact free boson of zero radius. Similarly, for $\tan\alpha = 1/k$, the deformed theory reduces to the vector coset (23) plus a free boson of zero radius. In fact one can easily see that all distinct deformations are parametrized by α in the range $-\pi/2$ to $\tan^{-1}(1/k)$. Any value of the parameter outside this range is equivalent to some value inside the range. Also, a study of the marginal deformations, as given in Eq. (17), shows that near the endpoints of the range of α the "speed" of marginal deformations becomes progressively smaller until it reaches zero at the end points which correspond to the axial and vector coset models. The $SU(2)$ cosets (plus free bosons) are therefore limit points on the marginal deformation line of the original $SU(2)_k$ WZNW model. This phenomenon has been interpreted as a continuous topology change in the context of the $SL(2,R)$ theory[17]. To understand why the coset models are obtained by fixed $O(2,2)$ transformations, we use Eq. (21) which states that the gauged model is independent of the parameter α. We can, therefore, start from a theory with any value of the parameter before introducing the gauge fields. In particular, we can look for some

$\alpha = \alpha_0$ for which one of the coordinates decouples as a free field. For example, in our case we can choose $\alpha = -\pi/2$ for which $\widetilde{\theta}$ decouples as a free field:

$$S_{(\alpha=\alpha_0)}(\phi,\theta,\widetilde{\theta}) = S'_{(\alpha=\alpha_0)}(\phi,\theta) + \frac{1}{2\pi}\int d^2z\bar{\partial}\widetilde{\theta}\partial\widetilde{\theta} \tag{24}$$

Now, we consider the axial $U(1)$ gauging of the above model. On the right hand side of the above equation, only the term containing $\widetilde{\theta}$ gets affected by gauging. Gauge fixing and integrating out the gauge field simply eliminates this term leaving $S'_{(\alpha=\alpha_0)}(\phi,\theta)$ unchanged. On the other hand, by Eq. (21), the left hand side in Eq. (24) after gauging must be the same as $S^{\rm gauged}_{(\alpha=0)}(\phi,\theta,\widetilde{\theta},A')$ which leads to the axial coset model. This argument shows that once we know that $S_{\alpha=\alpha_0}$ can be written in the form of Eq. (24), then $S'_{\alpha=\alpha_0}(\phi,\theta)$ must be the action of the gauged WZNW model (in our case, the $SU(2)/U(1)$ coset model). The vector gauging can also be understood along the same lines. Though the situation considered by Kumar [16] is more general, the results can be understood in the same way. Note that the above construction requires that the $O(d,d)$ transformed theory is written in a form that can be gauged by covariant derivative replacement.

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5. References

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