QFT problem set 2

Due date: Thursday, Dec 21, 10, 2023

1. In Dirac theory, starting from the expression for the Noether charges, express the Hamiltonian H, momentum P_i, $U(1)$ charge Q, and spin $\vec{S}$ operators in terms of the Dirac field and its conjugate.

   Compute Q in terms of the creation and annihilation operators (*4 points*).

2. In the Dirac theory, consider a 1-particle state $|\mathbf{1}_{p,r}\rangle$ of momentum vector $\vec{p}$ created by either $c_r^\dagger(\vec{p})$ or $d_r^\dagger(\vec{p})$. The projection of the spin operator $\vec{S}$ along the particle momentum, $S_p = \vec{S}\cdot\hat{p}$, is given by (in units where $\hbar = 1$, and normal ordered)

$$S_p = \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{N}(\psi^\dagger \sigma_p \psi)$$

   (See Mandl and Shaw for notations and the identities to use in this problem). Show that in terms of the lowering and raising operators, one gets,

$$\langle \mathbf{1}_{p,r}|S_p|\mathbf{1}_{p,r}\rangle = \frac{1}{2}(-1)^{r+1}\langle \mathbf{1}_{p,r}|\Big(d_r^\dagger(\vec{p})d_r(\vec{p}) + c_r^\dagger(\vec{p})c_r(\vec{p})\Big)|\mathbf{1}_{p,r}\rangle\,.$$

   (Important: Note that $\vec{p}$ is the momentum of the state, not to be confused with the dummy momentum labels, say, $\vec{p}\,'$ and $\vec{p}\,''$, that would appear in the expansions of ψ and $\psi^\dagger$. The same holds for the indices r, r' and r''.)

   What are the spin eigenvalues when $|\mathbf{1}_{p,r}\rangle$ is either the 1-particle states $c_r^\dagger(\vec{p})|0\rangle$ or $d_r^\dagger(\vec{p})|0\rangle$ (for $r = 1,2$)? (*4 points*)

3. Using the expansion of the Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ in terms of creation and annihilation operators, compute the covariant anticommutation relations $\{\psi_\alpha^\pm(x), \overline{\psi}_\beta^\mp(y)\}$ showing that the answer is given by eqn (4.52b) of Mandl and Shaw (2^{nd} Edition). (*3 points*)

4. Using the result of question 4, compute the fermion propagator $\langle 0|\mathrm{T}\left(\psi(x)\overline{\psi}(y)\right)|0\rangle$.

   Show that the result is $iS_\mathrm{F}(x-y)$ given by the contour integral

$$iS_\mathrm{F}(x) = \frac{i\hbar}{(2\pi\hbar)^4}\int \mathrm{d}^3p \int_{C_\mathrm{F}} \mathrm{d}p^0\, e^{-\mathrm{i}px/\hbar}\, \frac{\not{p} + mc}{p^2 - m^2c^2}$$

   where C_F is the Feynman contour in the complex p^0 plane (provide your reasoning and calculation details). (*3 points*)