QFT problem set 2

Due date: Friday, Jan 29, 2021

1. (a) In Dirac theory, starting from the expression for the Noether charge, express the Hamiltonian H, momentum P_i, and angular momentum M_{ij} operators in terms of the Dirac field. Also, compute the conserved quantities H, P_i in terms of the creation and annihilation operators (*4 points*).

   (b) In the Dirac theory, consider the spin operator $\vec{S}$ acting on a state of momentum vector $\vec{p}$. Then the projection of the spin operator along the particle momentum is given by (in units where $\hbar = 1$)

$$S_p = \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{N}(\psi^\dagger \sigma_p \psi)$$

   (See Mandl and Shaw for notations). Note that here $\vec{p}$ denotes a given momentum not to be confused with the dummy momentum labels, say, $\vec{p}\,'$ and $\vec{p}\,''$, of the momentum sums. Show that in terms of lowering and raising operators this becomes,

$$S_p|\mathbf{1}_{p,r}\rangle = \left(\frac{1}{2}\sum_r (-1)^{r+1}\left(d_r^\dagger(\vec{p})d_r(\vec{p}) + c_r^\dagger(\vec{p})c_r(\vec{p})\right)\right)|\mathbf{1}_{p,r}\rangle\,.$$

   (Hint: The term in S_p that contains a product of two creation operators does not contribute. You can assume that the coefficient of this term vanishes, but you need not show this here).

   What are the spin eigenvalues when $|\mathbf{1}_{p,r}\rangle$ is either of the 1-particle states $c_r^\dagger(\vec{p})|0\rangle$ or $d_r^\dagger(\vec{p})|0\rangle$ (for $r = 1, 2$)? (*4 points*)

2. Using the expansion of the Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ in terms of creation and annihilation operators, compute the covariant anticommutation relations $\{\psi_\alpha^\pm(x), \bar{\psi}_\beta^\mp(y)\}$ showing that the answer is given by eqn (4.52b) of Mandl and Shaw (2^{nd} Edition). (*3 points*)

3. Using the result of question 2, compute the fermion propagator $\langle 0|\mathrm{T}\left(\psi(x)\bar{\psi}(y)\right)|0\rangle$. Show that the result is $iS_\mathrm{F}(x-y)$ given by the contour integral

$$iS_\mathrm{F}(x) = \frac{i\hbar}{(2\pi\hbar)^4}\int \mathrm{d}^3p \int_{C_\mathrm{F}} \mathrm{d}p^0\, e^{-\mathrm{i}px/\hbar}\frac{\not{p} + mc}{p^2 - m^2c^2}$$

   where C_F is the Feynman contour in the complex p^0 plane (provide your reasoning and calculation details). (*3 points*)