QFT problem set 1

Due date: Friday Dec 01, 2023

(Use the metric $\eta_{00} = 1 = -\eta_{ii}$ and set $c = 1$)

1. The Lagrangian density for a free real scalar field $\phi(x)$ is $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$. Obtain the equation of motion for ϕ. Starting with a trial solution $\phi \sim e^{ik_\mu x^\mu}$ solve this equation and obtain the general solution (up to normalization)

$$\phi(x) = \sum_{\mathbf{k}} \left(\frac{1}{2V\omega_{\mathbf{k}}}\right)^{1/2} \left(a(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\right),$$

   where $kx = k_\mu x^\mu$ and $k^0 = \omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$, taking $\hbar = c = 1$. *(4 points)*

2. Derive the following expression for the lowering operator $a(\mathbf{k})$

$$a(\mathbf{k}) = \frac{1}{(2V\omega_{\mathbf{k}})^{1/2}} \int \mathrm{d}^3\mathrm{x}\,\mathrm{e}^{\mathrm{i}kx}\left(\mathrm{i}\dot{\phi}(x) + \omega_{\mathbf{k}}\phi(x)\right)$$

   and obtain the commutation relations for $a(\mathbf{k})$ and $a^\dagger(\mathbf{k}')$ using the equal time commutation relations for the fields (note the relation between $\dot{\phi}$ and π). *(4 points)*

3. Assume that a and $a^\dagger$ are lowering and raising operators satisfying $[a, a^\dagger] = 1$, $[a, a] = 0$ and $[a^\dagger, a^\dagger] = 0$. Given a state

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle\,,$$

   show that $\langle n|n\rangle = 1$. Furthermore, given $|n+1\rangle = C\,a^\dagger\,|n\rangle$, find C so that $|n+1\rangle$ is normalized to unity. *(4 points)*

   (Hint: It is convenient (but not necessary) to first evaluate the commutator $[a, (a^\dagger)^n]$.)

4. For the real scalar field, obtain the Hamiltonian H and momenta P_i in terms of $\dot{\phi}$ and $\vec{\nabla}\phi$. Using the solution for $\phi(x)$, calculate these in terms of the creation and annihilation operators, showing the steps in your calculation. *(4 points)*.