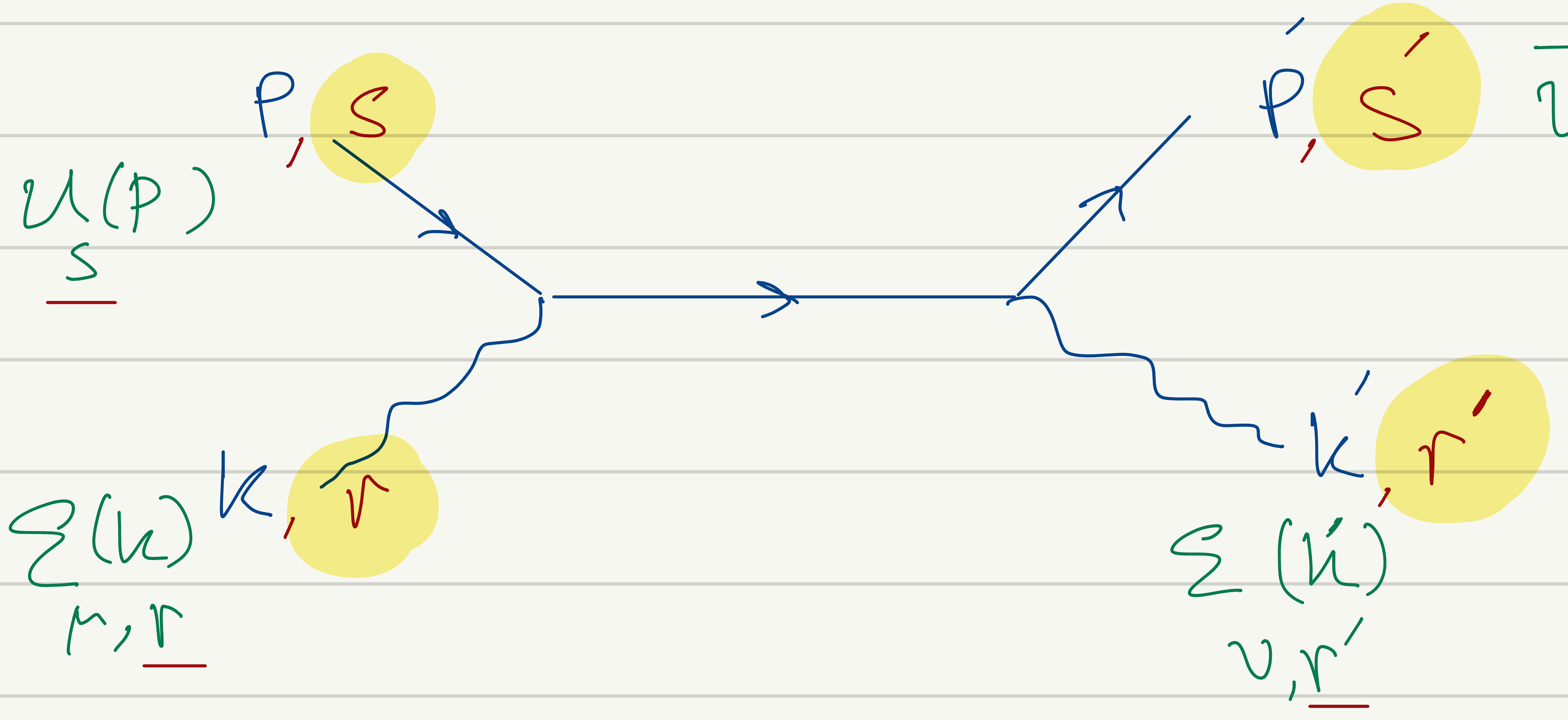
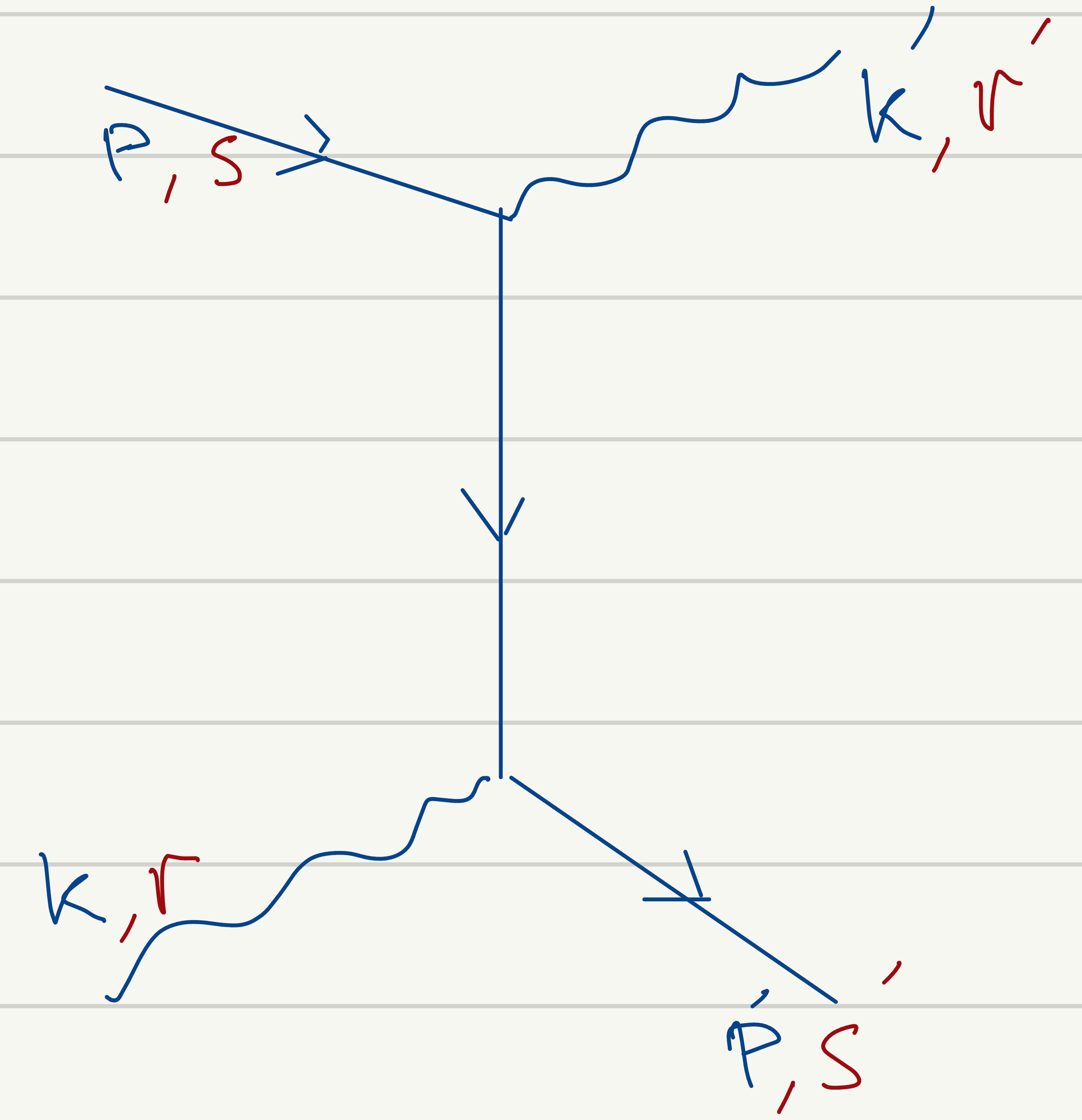
Polarized cross-sections:

Specify the spins and polarizations of the particles involved.

Example:



+



$$\mathcal{M}^{(a)}_{s s' r r'} = -e^2 \, \bar{u}_s(p') \, \not{\varepsilon}_{r'}(k') \, i S_F(q = p+k) \, \not{\varepsilon}_{r}(k) \, u_{s}(p)$$

$$\mathcal{M}^{(b)}_{s s' r r'} = \cdots\cdots$$

$$\left[\begin{array}{ll} s, s' = 1, 2 & (\text{spin } +\tfrac{1}{2}, -\tfrac{1}{2}) \\ r, r' = 1, 2 & (\text{two polarization states}). \end{array} \right]$$

$$\sigma_{s s' r r'} \sim \left| \mathcal{M}_{s s' r r'} \right|^2$$

- The cross-section depends on the specified spins & polarizations.

Unpolarized cross-section:

Allows for all possible spins and polarizations in $|i\rangle$ & $|f\rangle$

Example: compute σ irrespective of the values of spins and polarizations:

$$\sigma = \overline{\sum_i} \sum_f \sigma_{s,s',r,r'}$$

$\therefore$ Instead of $|\mathcal{M}|^2$, use $X = \overline{\sum_i} \sum_f |\mathcal{M}|^2$

- $\sum_f$: sum over final spins/polarizations.
- $\overline{\sum_i}$: average over initial spins/polarizations.

$$\overline{\sum_i} = \frac{1}{(\#\text{ of states})} \sum_i \qquad \left(\text{for } s=1,2 \text{ "\# of states"} = 2\right)$$

Digression : Energy projection operators

$$\sum_{r=1}^{2} u_{r\alpha}(p)\,\bar{u}_{r\beta}(p) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \equiv \Lambda^{+}_{\alpha\beta}(p).$$

$$-\sum_{r=1}^{2} v_{r\alpha}(p)\,\bar{v}_{r\beta}(p) = -\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} \equiv \Lambda^{-}_{\alpha\beta}(p)$$

$$\not{p}_{\alpha\beta} = p_\mu \gamma^\mu_{\alpha\beta}$$

$$p_\mu p^\mu = m^2$$

$$\not{p}\not{p} = p_\mu p^\mu$$

Check :

$$\Lambda^{+2} = \left(\frac{\not{p}+m}{2m}\right)\left(\frac{\not{p}+m}{2m}\right) = \frac{m^2+2m\not{p}+m^2}{(2m)^2} = \frac{2m(\not{p}+m)}{(2m)^2} = \frac{\not{p}+m}{2m} = \Lambda^+$$

$\therefore$

$$(\Lambda^+)^2 = \Lambda^+, \quad (\Lambda^-)^2 = \Lambda^-, \quad \Lambda^+ + \Lambda^- = \mathbb{1}$$

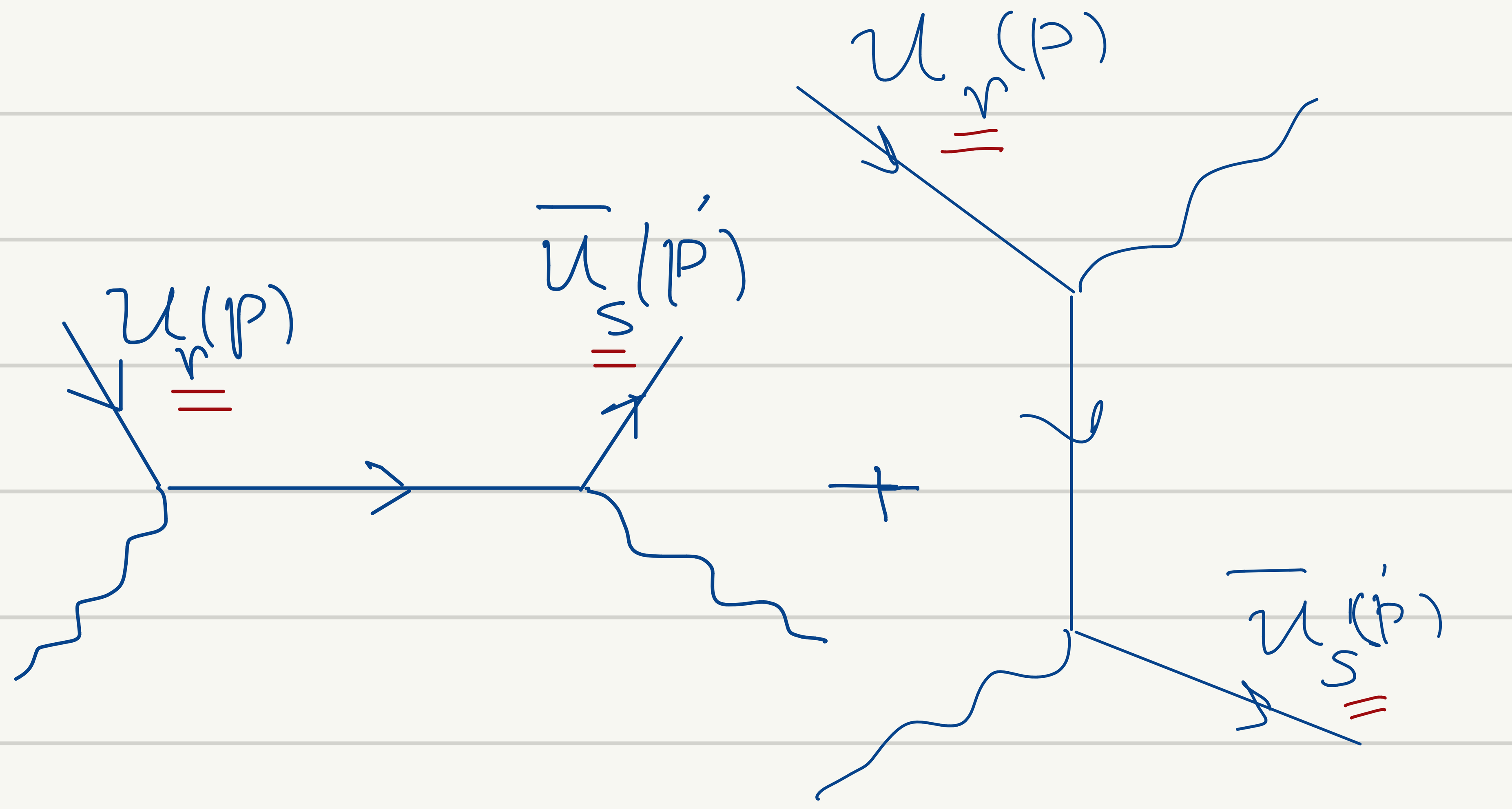
(projection operators.)

Spin sums

$$|\mathcal{M}|^2 \longrightarrow \quad X = \sum_i \sum_f |\mathcal{M}|^2$$

Example: $\mathcal{M} = \bar{u}_{\underline{\underline{s}}}(p')\, \Gamma\, u_{\underline{\underline{r}}}(p)$

final spin (s) initial spin (r)



$$X = \overline{\sum_i} \sum_f |\mathcal{M}|^2 = \left(\frac{1}{2} \sum_{r=1}^{2}\right) \sum_{s=1}^{2} \mathcal{M}\mathcal{M}^*$$

$$\mathcal{M}^* = \mathcal{M}^{*T} = \mathcal{M}^\dagger = \left(\bar{u}_s(p')\,\Gamma\, u_r(p)\right)^\dagger = \left(u_s^\dagger(p')\,\gamma^0\,\Gamma\, u_r(p)\right)^\dagger$$

$$= u_r^\dagger(p)\,(\gamma^0\gamma^0)\,\Gamma^\dagger\gamma^0 u_s(p') = \bar{u}_r(p)\,\tilde{\Gamma}\, u_s(p')$$

$$\boxed{\mathcal{M}^* = \bar{u}_r(p)\,\tilde{\Gamma}\, u_s(p')}$$

$$\left(\gamma^0\gamma^0 = 1, \quad \tilde{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0\right)$$

$$X = \overline{\sum_i}\sum_f |\mathcal{M}|^2 = \overline{\sum_i}\sum_f \mathcal{M}\mathcal{M}^\dagger = \frac{1}{2}\sum_r\sum_s \left(\bar{u}_s(p')\,\Gamma\, u_r(p)\right)\left(\bar{u}_r(p)\,\tilde{\Gamma}\, u_s(p')\right)$$

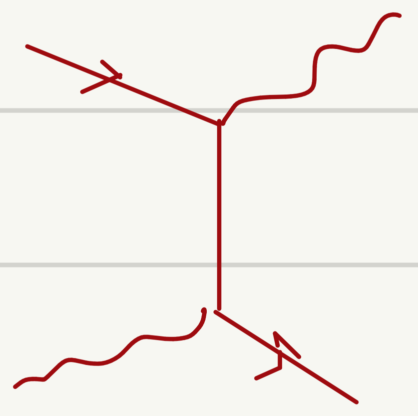
$$= \frac{1}{2}\sum_{\alpha,\beta,\rho,\sigma}\sum_r\sum_s \left(\bar{u}_{s\alpha}\Gamma_{\alpha\beta}u_{r\beta}\right)\left(\bar{u}_{r\rho}\tilde{\Gamma}_{\rho\sigma}u_{s\sigma}\right) = \frac{1}{2}\Big(\underbrace{\sum_s u_{s\sigma}\bar{u}_{s\alpha}}_{\Lambda^+_{\sigma\alpha}(p')}\Big)\Gamma_{\alpha\beta}\Big(\underbrace{\sum_r u_{r\beta}\bar{u}_{r\rho}}_{\Lambda^+_{\beta\rho}(p)}\Big)\tilde{\Gamma}_{\rho\sigma}$$

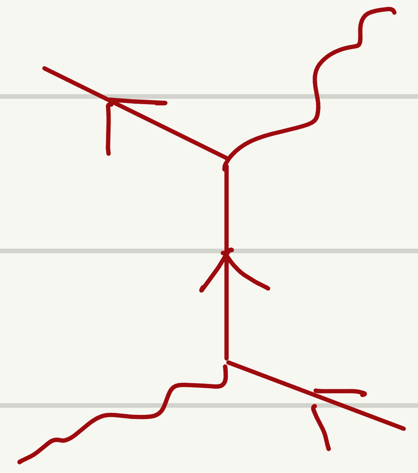
$$= \frac{1}{2}\sum_{\alpha,\beta,\sigma,\rho}\Lambda^+_{\sigma\alpha}(p')\,\Gamma_{\alpha\beta}\,\Lambda^+_{\beta\sigma}(p)\,\tilde{\Gamma}_{\rho\sigma} = \boxed{\frac{1}{2}\,\mathrm{tr}\left(\Lambda^+(p')\,\Gamma\,\Lambda^+(p)\,\tilde{\Gamma}\right) = X}$$

$$\frac{\not{p}'+m}{2m} \qquad \frac{\not{p}+m}{2m}$$

In $\overline{\sum_i}\sum_f |\mathcal{M}|^2$ the spinors $u_r(p), \bar{u}_r(p), v_r(p), \bar{v}_r(p)$ always appear in the combinations $\Lambda^+(p)$ and $\Lambda^-(p)$.

In general $\mathcal{M}$ contains one or more of the following structures:

A) $\bar{u}_s(p')\,\Gamma\,u_r(p)$ 

C) $\bar{v}_s(p')\,\Gamma\,v_r(p)$ 

B) $\bar{u}_s(p')\,\Gamma\,v_r(p)$ 

D) $\bar{v}_s(p')\,\Gamma\,u_r(p)$ 

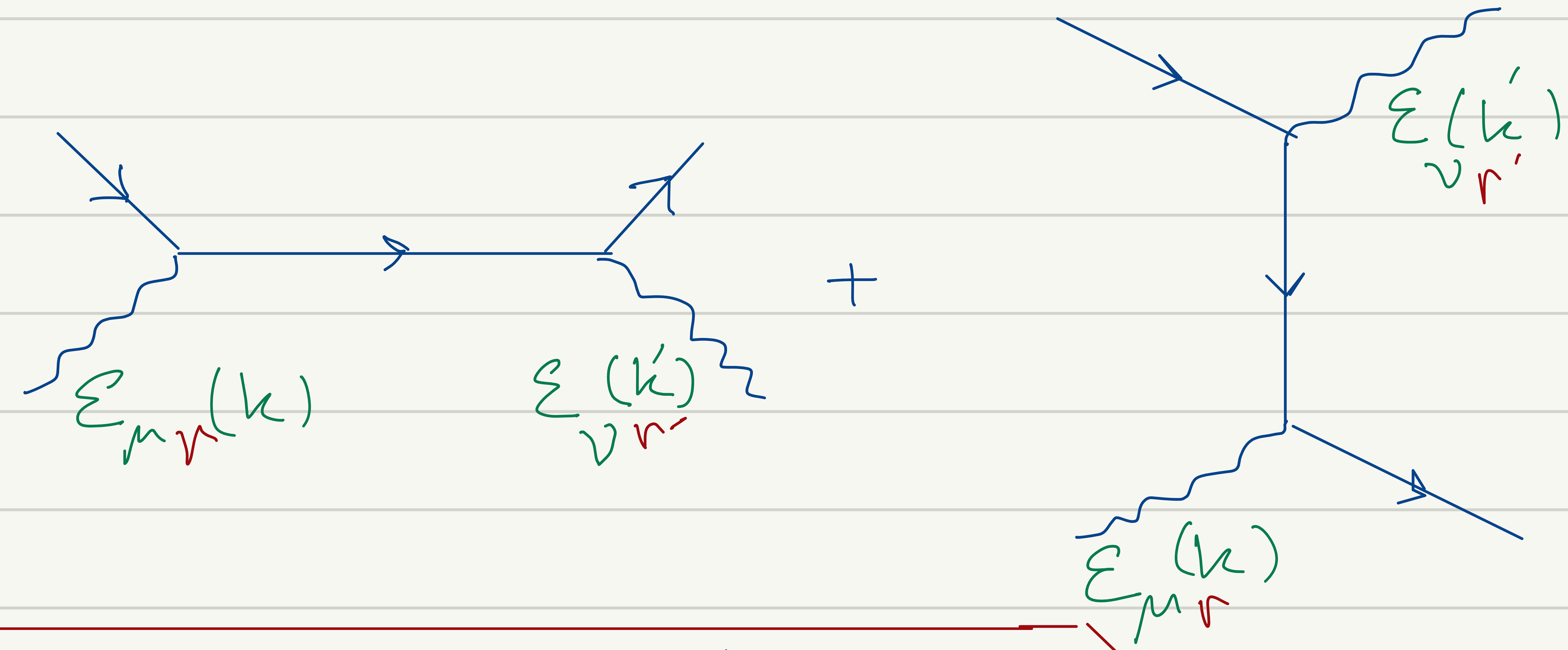
Example:

B) $$X = \sum_r \sum_s |\mathcal{M}|^2 = -\,\mathrm{tr}\left(\frac{\not{p}'+m}{2m}\,\Gamma\,\frac{\not{p}-m}{2m}\,\tilde{\Gamma}\right)$$

Note: there is no factor of $\frac{1}{2}$ since both fermions are in final state.

Polarization sums:

$$\mathcal{M} = \varepsilon_{\mu,r}(k) \boxed{\varepsilon_{\nu,r'}(k') \, \mathcal{M}^{\mu\nu}}$$



$$\mathcal{M} = \varepsilon_{\mu,r}(k) \, \mathcal{M}^{\mu}$$

the same procedure is also applied to $\varepsilon_{\nu,r'}(k')$ which is suppressed here

$$\mathcal{M}^* = \varepsilon_{\mu,r}(k) \, \mathcal{M}^{*\mu}$$

$$\sum_r |\mathcal{M}|^2 = \sum_{r=1}^{2} \varepsilon_{\mu,r}(k) \, \varepsilon_{\nu,r}(k) \, \mathcal{M}^{\mu}(k) \, \mathcal{M}^{*\nu}(k)$$

$$= -\eta_{\mu\nu} \, \mathcal{M}^{\mu} \mathcal{M}^{*\nu} = \mathcal{M}^{\mu} \mathcal{M}^{*}_{\mu}$$

Prescription: replace $\boxed{\sum_{r=1}^{2} \varepsilon_{\mu r}(k) \, \varepsilon_{\nu r}(k) \longrightarrow -\eta_{\mu\nu}}$ Why?

Recall:

$$\sum_{s,r=0}^{3} \eta^{rs} \varepsilon_{\mu r} \varepsilon_{\nu s} = \eta_{\mu\nu} \qquad \eta^{rs} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\varepsilon_{\mu 0} \varepsilon_{\nu 0} - \varepsilon_{\mu 1} \varepsilon_{\nu 1} - \varepsilon_{\mu 2} \varepsilon_{\nu 2} - \varepsilon_{\mu 3} \varepsilon_{\nu 3} = \eta_{\mu\nu}$$

$$\therefore \sum_{r=1}^{2} \varepsilon_{\mu,r}(k)\, \varepsilon_{\nu,r}(k) = -\eta_{\mu\nu} + \left(\varepsilon_{\mu 0}\varepsilon_{\nu 0} - \varepsilon_{\mu 3}\varepsilon_{\nu 3}\right)$$

$$\boxed{\sum_{r=1}^{2} \varepsilon_{\mu,r}(k)\, \varepsilon_{\nu,r}(k) = -\eta_{\mu\nu} - \frac{1}{(kn)^2}\left[k^\mu k^\nu - (kn)\left(k^\mu n^\nu + k^\nu n^\mu\right)\right]}$$

(using $\varepsilon_{\mu 0}$ & $\varepsilon_{\mu 3}$, where $n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$)

A gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta \Rightarrow \varepsilon_{\mu,r}(k) \to \varepsilon_{\mu,r}(k) + \lambda_r k_\mu$

(see next pages)

Since QED is gauge invariant,

$$\mathcal{M} = \varepsilon_{\mu,r}(k)\, \mathcal{M}^\mu = \left(\varepsilon_{\mu,r}(k) + \lambda_r k_\mu\right) \mathcal{M}^\mu \Rightarrow \boxed{k_\mu \mathcal{M}^\mu = 0}$$

Hence in the expression for $\sum_{r=1}^{2} \varepsilon_{\mu r}(k)\, \varepsilon_{\nu r}(k)$, all terms with at least one factor of k_μ or k_ν drop out of $\sum_{r=1}^{2} \varepsilon_{\mu\nu}\, \varepsilon_{\nu r}\, \mathcal{M}^\mu \mathcal{M}^{*\nu}$, i.e.,

$$\sum_{r=1}^{2} |\mathcal{M}|^2 = \left(-\eta_{\mu\nu} - \frac{1}{(kn)^2} \left[\cancel{k^\mu k^\nu} - (kn)\cancel{(k^\mu n^\nu} + \cancel{k^\nu n^\mu)} \right] \right) \mathcal{M}^\mu \mathcal{M}^{*\nu}$$

$$\therefore \quad \boxed{\sum_{r=1}^{2} \varepsilon_{\mu r}(k)\, \varepsilon_{\nu r}(k)\, \mathcal{M}^\mu \mathcal{M}^{*\nu} = -\eta_{\mu\nu} \mathcal{M}^\mu \mathcal{M}^{*\nu}}$$

Gauge transformations of $\varepsilon_{\mu,r}(k)$:

We quantized electromagnetism in the Lorenz gauge $\partial_\mu A^\mu = 0$

Residual gauge invariance of QED in Lorenz gauge:

$A'_\mu = A_\mu + \partial_\mu f(x)$, for $\Box f(x) = 0$ (so that $\partial_\mu A^\mu = 0 \Rightarrow \partial_\mu A'^\mu = 0$)

$$\underline{A_\mu} = \sum_{\vec{k},r} (\ast) \left(\varepsilon_{\mu r}(k)\, a_r(k)\, e^{-ikx} + \varepsilon_{\mu r}(k)\, a^\dagger_r(k)\, e^{ikx} \right)$$

$$\underline{A'_\mu} = \sum_{\vec{k},r} (\ast) \left(\varepsilon'_{\mu r}(k)\, a_r(k)\, e^{-ikx} + \varepsilon'_{\mu r}(k)\, a^\dagger_r(k)\, e^{ikx} \right)$$

$\Rightarrow A'_\mu - A_\mu = \sum \ast \left(\varepsilon'_{\mu r} - \varepsilon_{\mu r}\right) a_r(k) \ldots$

$\Rightarrow$

$$\underline{A'_\mu - A_\mu} = \sum_{k,r} (\ast) \left[\left(\varepsilon'_{\mu r} - \varepsilon_{\mu r}\right) a_r(k)\, e^{-ikx} + \left(\varepsilon'_{\mu r} - \varepsilon_{\mu r}\right) a^\dagger_r(k)\, e^{ikx} \right]$$

$$\Box f = 0 \Rightarrow \quad f(x) = \sum_{\vec{k}} (\ast) \left(\lambda(\vec{k})\, e^{-ikx} + \lambda^\dagger(\vec{k})\, e^{ikx} \right), \quad (k^2 = k_\mu k^\mu = 0)$$

$$\underline{\partial_\mu f} = \sum_k (\ast) \left(i k_\mu \lambda(k)\, e^{-ikx} + \left(i k_\mu \lambda(k)\right)^\dagger e^{ikx} \right) = A'_\mu - A_\mu$$

$$\Rightarrow \quad \left.\begin{aligned} i k_\mu \lambda &= \left(\mathcal{E}'_{\mu r}(k) - \mathcal{E}_{\mu r}(k)\right) a_r(k) \\ (i k_\mu \lambda)^\dagger &= \left(\mathcal{E}'_{\mu r}(k) - \mathcal{E}_{\mu r}(k)\right) a_r^\dagger(k) \end{aligned}\right\} \Rightarrow i\lambda(k) = \lambda_r(k)\, a_r(k)$$

$$\therefore \quad \mathcal{E}'_{\mu r}(k) - \mathcal{E}_{\mu r}(k) = k_\mu \lambda_r(k)$$

or

$$\boxed{\mathcal{E}'_{\mu r}(k) = \mathcal{E}_{\mu r}(k) + \lambda_r(k)\, k_\mu}$$

Note: Physics is not affected by gauge transformations.

$\therefore$ $k_\mu \mathcal{M}^\mu = 0$ is valid only when $\mathcal{M}$ is the complete amplitude, e.g. $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$ in compton scattering, but $k_\mu \mathcal{M}_a^\mu \neq 0$, $k_\mu \mathcal{M}_b^\mu \neq 0$.

You will use the spin sum and polarization sum techniques when computing unpolarized cross-sections in problem set 3.