Non-abelian gauge theories:

Recall:

Abelian gauge theory: $\psi \longrightarrow e^{i\theta}\psi = \psi'$

$\bar{\psi} \longrightarrow \bar{e}^{i\theta}\psi = \bar{\psi}'$

Global case $\partial_\mu \theta = 0$

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$

is invariant.

Local case $\partial_\mu \theta \neq 0$:

Minimal substitution: $\partial_\mu \psi \longrightarrow D_\mu \psi = (\partial_\mu + iq A_\mu)\psi$

$$D'_\mu \psi' = e^{i\theta(x)} D_\mu \psi \quad , \quad A'_\mu = A_\mu - \partial_\mu \theta$$

U(1) gauge invariant Lagrangian:

$$\mathcal{L} = \bar{\psi}(i D\!\!\!\!/ - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \underbrace{q(\bar{\psi}\gamma^\mu\psi)}_{J^\mu} A_\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \qquad (q = -e)$$

The Gauge Principle:

Starting from a theory of free (non-interacting) fields with a continuous global symmetry, the gauge principle allows us to construct an interacting theory by:

1) making the transformations local (this is not a symmetry of the free theory)
2) restoring the symmetry by introducing gauge fields in the theory through covariant derivatives
3) writing an action for the gauge fields.

The gauge fields then give rise to interactions between the original fields and among themselves. The electromagnetic, weak nuclear, and strong nuclear forces are all produced by gauge fields in this way. The resulting quantum field theories are renormalizable because of the local gauge invariance.

Non-Abelian gauge theories are based on non-Abelian symmetry groups.

Non-abelian theory

Lagrangian for n fermions ψ_a $(a=1,\dots,n)$, $\{\psi_{a\alpha}\}$

$$\mathcal{L} = \sum_{a=1}^{n} \left(i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a - m_a \bar{\psi}_a \psi_a \right)$$

Set $m_1 = m_2 = \cdots = m_n = m$ and define $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$

$$\boxed{\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

where,

$$\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, \dots \bar{\psi}_n)$$

Symmetry: $\Psi' = U\Psi$, $\bar{\Psi}' = \bar{\Psi} U^\dagger$ U: $n \times n$ matrices
$U^\dagger U = \mathbb{1}$

SU(n) symmetry:

- U : $n \times n$ matrix, $\quad UU^\dagger = U^\dagger U = \mathbb{1}$

  $\Rightarrow$ set of all unitary matrices $\quad U(n)$

- $\det U = 1$: special unitary matrices $\quad SU(n)$

$$\Psi' = U\Psi = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}\begin{pmatrix} \vdots \end{pmatrix}, \qquad \psi'_a = U_{ab}\psi_b$$

$$\bar{\Psi}' = \Psi'^\dagger \gamma^0 = \Psi^\dagger U^\dagger \gamma^0 = \Psi^\dagger \gamma^0 U^\dagger = \bar{\Psi} U^\dagger$$

If $\partial_\mu (U_{ab}) = 0$, $\quad \partial_\mu \Psi' = \partial_\mu (U\Psi) = U(\partial_\mu \Psi)$

Then, $\quad \bar{\Psi}'\Psi' = \bar{\Psi} U^\dagger U \Psi = \bar{\Psi}\Psi \quad$ similarly, $\bar{\Psi}'\partial_\mu \Psi' = \bar{\Psi}\partial_\mu \Psi$

Digression: The set of all SU(n) matrices form the SU(n) group.

Definition: A group G is a set of elements $\{G_1, G_2, G_3, \cdots\}$ with a composition rule $G_1 \circ G_2$ and the following 4 properties:

1. For any $G_1 \in G$ and $G_2 \in G$, one always has $G_1 \circ G_2 \in G$.
2. Associativity: $G_1 \circ (G_2 \circ G_3) = (G_1 \circ G_2) \circ G_3$
3. Existence of an identity element $\mathbb{1}$ such that
$$\mathbb{1} \circ G_i = G_i \circ \mathbb{1} = G_i \quad \text{for any element } G_i$$
4. Existence of an inverse G_i^{-1} for every G_i such that
$$G_i \circ G_i^{-1} = G_i^{-1} G_i = \mathbb{1}$$

SU(n) is a continuous group.

SU(n) matrices :

Def: Let U : $n \times n$ matrix, $U_{ab} \in \mathbb{C}$, $(a, b = 1, \dots, n)$

$$\boxed{U^\dagger U = U U^\dagger = \mathbb{1}, \qquad \det U = 1}$$

Number of free parameters in U :

$$\underbrace{2n^2}_{U} - \underbrace{n^2}_{U^\dagger U = 1} - \underbrace{1}_{\det U = 1} = n^2 - 1$$

$\ln U = iT + (\#)$, $e^{(\#)} = 1$

Exponential parametrization: Let $\boxed{U = e^{iT}}$ T : $n \times n$ matrix

Then

- $U^\dagger = U^{-1} \Rightarrow e^{-iT^\dagger} = e^{-iT} \Rightarrow \boxed{T^\dagger = T}$
- $\underbrace{\det U \equiv e^{\operatorname{tr}(\ln U)}}_{\text{(relation used)}} = e^{i \operatorname{tr} T} = 1 \Rightarrow \boxed{\operatorname{tr}(T) = 0}$

No of free parameters in T ($T_{ab} \in \mathbb{C}$) :

$$\underbrace{2n^2}_{T} - \underbrace{n^2}_{T=T^\dagger} - \underbrace{1}_{\mathrm{tr}\,T=0} = n^2-1 \quad (\text{same as in } U)$$

We can write: $\quad T = \sum_{j=1}^{n^2-1} \theta_j T_j$, where,

θ_j : n^2-1 real free parameters $(j=1, \ldots, n^2-1)$

T_j : fixed $n\times n$ matrices providing a basis in the space of Hermitian traceless matrices.

Then:

$$\boxed{U = e^{i\sum_{j=1}^{n^2-1} \theta_j T_j}} \quad , \quad \boxed{[T_i, T_j] = i f_{ijk} T_k}$$

"Lie algebra" of $SU(n)$

T_j : generators of $SU(n)$ $\left(T_j^\dagger = T_j \, , \quad \mathrm{tr}(T_j) = 0 \right)$

f_{ijk} : antisymmetric in "i,j,k", fixed numbers

→ structure constants of $SU(n)$

Any $T = T^\dagger$, $\mathrm{tr}\, T = 0$ can be expanded in the basis of T_j.

(T_j : basis in the space of Hermitian, traceless matrices)

For infinitesimal $\theta_j \ll 1$, $\boxed{U = \mathbb{1} + i \sum_{j=1}^{n^2-1} \theta_j T_j + \mathcal{O}(\theta^2)}$

Normalization : $\boxed{\mathrm{tr}\,(T_i T_j) = \frac{\delta_{ij}}{2}}$

Example: $SU(2)$: 2×2, complex matrices, $n^2-1=3$

$$U_{2\times 2} = e^{i\sum_{j=1}^{3}\theta_j\left(\frac{\sigma_j}{2}\right)}$$

Generators: $T_j = \frac{\sigma_j}{2}$, σ_j : 2×2 Pauli matrices

$$\boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Structure constants: $f_{ijk} = \epsilon_{ijk}$

$$\left(\epsilon_{123} = 1\right)$$

SU(n) transformations:

$$\Psi \longrightarrow \Psi' = U\,\Psi \qquad \left(\Psi'_a = \sum_{b=1}^{n} U_{ab}\,\psi_b \right)$$

$$U \in SU(n), \qquad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

$$\boxed{\text{Global } SU(n): \ \partial_\mu U_{ab} = 0}$$

SU(n) invariant Lagrangians

Global SU(n): $\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = \sum_{a=1}^{n} \bar{\psi}_a (i\gamma^\mu \partial_\mu - m)\psi_a$

Infinitesimal form: $\Psi' = \left(\mathbb{1} + i\sum_{j=1}^{n^2-1} \theta_j T_j\right)\Psi \Rightarrow \boxed{\delta\Psi = i\sum_{j=1}^{n^2-1} \theta_j T_j \Psi}$

or, $\delta\psi_a = i\sum_j \theta_j (T_j)_{ab}\,\psi_b$

Conserved currents:

$$J^\mu_{(\theta)} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_a} \delta\psi_a = i\bar{\psi}_a \gamma^\mu (\delta\psi_a)$$

$$= -\sum_j \theta_j \bar{\psi}_a \gamma^\mu (T_j)_{ab} \psi_b$$

$$= -\sum_j \theta_j J^\mu_j$$

$$\boxed{J^\mu_i = \bar{\psi}_a \gamma^\mu (T_i)_{ab} \psi_b \equiv \bar{\Psi} \gamma^\mu T_i \Psi} \qquad \partial_\mu J^\mu_i = 0, \quad (i = 1, \dots, n^2-1)$$

Conserved charges:

$$Q_i = \int d^3x \, J^0_i = \int d^3x \left(\Psi^\dagger T_j \Psi \right)$$

Using the free field expansions for the ψ_a, charges Q_i can be expressed in terms of creation & annihilation operators $c_a, c_a^\dagger, d_a, d_a^\dagger$.

One finds:

$$\boxed{[Q_i, Q_j] = i f_{ijk} Q_k}$$

($SU(n)$ Lie algebra similar to the T_i)

T_i : $n \times n$ matrices that generate the $SU(n)$ transformation on the n-component vectors, $\psi_a(x)$ as

$$\boxed{U_{ab} \psi_b = \left(e^{i \sum_j \theta_j T_j}\right)_{ab} \psi_b = \psi_a'}$$

Q_i : operators that generate the $SU(n)$ transformations on the Hilbert space of states in QFT:

$$\boxed{\hat{U} |s\rangle = \left(e^{i \sum_{i=1}^{n^2-1} \theta_j Q_j}\right) |s\rangle = |s'\rangle}$$

Local SU(n) gauge invariance :

Consider $\partial_\mu U_{ab} \neq 0 \Rightarrow \theta_j \rightarrow g\,\theta_j(x)$ $\quad$ (g = constant)

Then,
$$\boxed{U(x) = e^{ig\sum_{i=1}^{n^2-1} \theta_j(x) T_j}}$$

$\boxed{U^\dagger = U^{-1}}$

- $\Psi' = U(x)\,\Psi$, $\quad \bar{\Psi}' = \bar{\Psi}\, U^\dagger \Rightarrow \bar{\Psi}'\Psi' = \bar{\Psi}\Psi$ (invariant)
- $\partial_\mu \Psi' = \partial_\mu (U(x)\,\Psi) = U(x)\,\partial_\mu \Psi + (\partial_\mu U)\Psi$

$$\Rightarrow \bar{\Psi}'\gamma^\mu \partial_\mu \Psi' = \bar{\Psi}\gamma^\mu \partial_\mu \Psi + \underline{\bar{\Psi}\gamma^\mu (U^{-1}\partial_\mu U)\Psi}$$ (not invariant)

Replace $\partial_\mu \Psi$ by a covariant derivative $D_\mu \Psi$.

Covariant derivative:

Require that if $\Psi' = U(x)\Psi$, then $(D'_\mu \Psi') = U(x)(D_\mu \Psi)$

$\Rightarrow \quad i\bar{\Psi}'\gamma^\mu D'_\mu \Psi' = i\bar{\Psi}\gamma^\mu D_\mu \Psi \quad$ (invariant)

Minimal substitution:

what is $D_\mu \Psi$?

$\boxed{U^{-1} = U^\dagger}$

Define $\boxed{D_\mu \Psi = (\partial_\mu + igA_\mu)\Psi}$

The transformation of iA_μ needs to cancel the $U^{-1}\partial_\mu U$ term.

- $iU^{-1}\partial_\mu U$ is hermitian: $(iU^{-1}\partial_\mu U)^\dagger = -i\,\partial_\mu U^\dagger U = -i[\partial_\mu(\overset{\nearrow 0}{U^\dagger U}) - U^\dagger \partial_\mu U] = iU^{-1}\partial_\mu U$.
- $iU^{-1}\partial_\mu U$ is traceless: Use $\boxed{\partial_\mu(\det U) = (\det U)\,\mathrm{tr}(U^{-1}\partial_\mu U)}$ (derivation on next page)

since $\det U = 1$, $\partial_\mu(1) = 0 \Rightarrow \mathrm{tr}(U^{-1}\partial_\mu U) = 0$

Digression: How to compute $\partial_\mu(\det U)$?

Start with $\det U = e^{\mathrm{tr}\ln U}$

Then, $\det(U+\delta U) = e^{\mathrm{tr}\ln(U+\delta U)} = e^{\mathrm{tr}\ln[U(1+U^{-1}\delta U)]}$

$$= e^{\mathrm{tr}[\ln U + \ln(1+U^{-1}\delta U)]}$$

$$= \left(e^{\mathrm{tr}\ln U}\right) e^{\mathrm{tr}\ln(1+U^{-1}\delta U)}$$

$$= (\det U)\left(1 + \mathrm{tr}(U^{-1}\delta U) + \cdots\right)$$

$\ln(1+x) = x + \cdots$

$e^{\mathrm{tr}\,x} = 1 + \mathrm{tr}\,x + \cdots$

$$\delta(\det U) = \det(U+\delta U) - \det U = \mathrm{tr}(U^{-1}\delta U) + \cdots$$

$$\Rightarrow \quad \boxed{\partial_\mu(\det U) = \mathrm{tr}(U^{-1}\partial_\mu U) = -\mathrm{tr}(U\partial_\mu U^{-1})}$$

$\left(\text{since } \partial_\mu(U^{-1}U) = 0\right)$

$\therefore$ $A_\mu(x)$ is a hermitian and traceless $n \times n$ matrix $(A_\mu)_{ab}$

(like T in $U = e^{iT}$)

Similar to T, $A_\mu(x)$ can be expanded in a basis of hermitian, traceless matrices $\{T^i\}$:

$$\boxed{A_\mu = \sum_{i=1}^{n^2-1} A_\mu^i(x)\, T^i}$$

$A_\mu^i(x)$: n^2-1 vector fields called gauge fields (or Yang-Mills fields)

To do: 1) Find the transformation properties of A_μ^i under U.

2) Find an action/Lagrangian for $A_\mu^i(x)$.

Transformation of $A^i_\mu(x)$:

By definition: $\boxed{(D'_\mu \Psi') = U(D_\mu \Psi)}$

$$\Rightarrow \quad (\partial_\mu + igA'_\mu)U\Psi = U(\partial_\mu + igA_\mu)\Psi \qquad (\text{for any } \Psi)$$

$$\boxed{A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}}$$

The $U(1)$ case: $U = e^{ig\theta(x)}$, $\quad \underline{U A_\mu U^{-1}} = \cancel{e^{ig\theta}} A_\mu \cancel{e^{-ig\theta}} = A_\mu$

$$\underline{U^{-1}\partial_\mu U} = \cancel{U^{-1}}(ig\partial_\mu\theta \cancel{U}) = ig\partial_\mu\theta \Rightarrow \boxed{A'_\mu = A_\mu - \partial_\mu\theta}$$

Extracting A^i_μ from the matrix A_μ:

$$\mathrm{tr}(A_\mu T^i) = \mathrm{tr}(A^j_\mu T^j T^i) = A^j_\mu \,\mathrm{tr}(T^j T^i) = A^j_\mu \frac{\delta^{ji}}{2} = \frac{1}{2}A^i_\mu$$

$$\therefore \qquad \boxed{A^i_\mu(x) = 2\,\mathrm{tr}\left(A_\mu(x)\, T^i\right)}$$

Using this, the gauge transformation of A^i_μ can be worked out from that of A_μ. But a nice expression can be obtained only for infinitesimal transformations,

$$U = 1 + ig \sum_i \theta^i T^i + \mathcal{O}(\theta^2), \quad \text{to first order in } \theta^i$$

$$\underline{U A_\mu U^{-1}} = (1 + ig \sum \theta^i T^i)\, A^j_\mu T^j (1 - ig \sum \theta^k T^k) = A^j_\mu T^j + ig \sum_i \theta^i A^j_\mu (T^i T^j - T^j T^i)$$

$$= \left(A^k_\mu + ig A^j_\mu \theta^i (i f^{ijk})\right) T^k = \sum_k (A^k_\mu - g\theta^i f^{ijk} A^j_\mu).T^k$$

$$\underline{\frac{i}{g}(\partial_\mu U)\, U^{-1}} = \frac{i}{g}(ig) \sum_k \partial_\mu \theta^k T^k = -\sum_k \partial_\mu \theta^k T^k$$

$$\therefore \sum_k A'^k_\mu T^k = \sum_k \left(A^k_\mu - g\theta^i f^{ijk} A^j_\mu - \partial_\mu \theta^k\right) T^k$$

$$\Rightarrow \quad \boxed{\delta A^k_\mu = -\partial_\mu \theta^k - g\,\theta^i A^j_\mu f^{ijk}}$$

Gauge invariant Lagrangian so far:

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi = i\bar{\Psi}\gamma^\mu(\partial_\mu + igA_\mu)\Psi - m\bar{\Psi}\Psi$$

$$= \bar{\Psi}(i\not{\partial} - m)\Psi - g\bar{\Psi}\gamma^\mu A_\mu \Psi \qquad \left(A_\mu = \sum_{i=1}^{n^2-1} A^i_\mu T^i\right)$$

$$= \sum_{a=1}^{n} \bar{\psi}_a (i\not{\partial} - m)\psi_a - g\sum_{i=1}^{n^2-1}\sum_{a,b=1}^{n}\left(\bar{\psi}_a \gamma^\mu (T^i)_{ab}\psi_b\right) A^i_\mu$$

$$= \sum_{a=1}^{n} \bar{\psi}_a (i\not{\partial} - m)\psi_a - g\sum_{i=1}^{n^2-1} J^{\mu i} A^i_\mu$$

non-Abelian conserved currents

Action for $A^j_\mu(x)$:

Need to find a generalization of the fieldstrength tensor of Maxwell theory ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). Since $D_\mu \Psi$ transforms in the same way as Ψ, we have,

$$\left(D'_\mu D'_\nu \Psi'\right) = U\left(D_\mu D_\nu \Psi\right)$$

Evaluate: $D_\mu D_\nu \Psi - D_\nu D_\mu \Psi \equiv [D_\mu, D_\nu]\Psi$.

$$D_\mu D_\nu \Psi = (\partial_\mu + ig A_\mu)(\partial_\nu + igA_\nu)\Psi = (\partial_\mu + igA_\mu)(\partial_\nu \Psi + igA_\nu \Psi)$$

$$= \cancel{\partial_\mu \partial_\nu \Psi} + ig\, \partial_\mu A_\nu \Psi + \cancel{ig A_\nu \partial_\mu \Psi} + \cancel{ig A_\mu \partial_\nu \Psi} - g^2 A_\mu A_\nu \Psi.$$

$$D_\nu D_\mu \Psi = \cancel{\partial_\nu \partial_\mu \Psi} + ig\, \partial_\nu A_\mu \Psi + \cancel{ig A_\mu \partial_\nu \Psi} + \cancel{ig A_\nu \partial_\mu \Psi} - g^2 A_\nu A_\mu \Psi$$

$$\boxed{[D_\mu, D_\nu]\Psi = ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]\right)\Psi}$$

Defining $F_{\mu\nu}$ as $\quad [D_\mu, D_\nu]\Psi = ig F_{\mu\nu}\Psi$, we get

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]}$$

<u>Transformations:</u>

since: $D'_\mu D'_\nu \Psi' = U D_\mu D_\nu \Psi$, we have:

$$F'_{\mu\nu}\Psi' = \frac{1}{ig}(D'_\mu D'_\nu - D'_\nu D'_\mu)\Psi' = \frac{1}{ig} U (D_\mu D_\nu - D_\nu D_\mu)\Psi = U F_{\mu\nu}\Psi$$

Hence,

$$F'_{\mu\nu} U \Psi = U F_{\mu\nu}\Psi \qquad (\text{for every } \Psi)$$

$$\Rightarrow \quad \boxed{F'_{\mu\nu} = U F_{\mu\nu} U^{-1}}$$

$F_{\mu\nu}$ is a hermitian traceless matrix $\Rightarrow$ $F_{\mu\nu} = \sum_{i=1}^{n^2-1} F^i_{\mu\nu} T^i$.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$
$$= (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) T^i + ig\,[A^j_\mu T^j, A^k_\nu T^k]$$
$$= (\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) T^i + ig\,[T^j, T^k] A^j_\mu A^k_\nu$$
$$= \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig\,(if^{jki}) A^j_\mu A^k_\nu\right) T^i$$
$$= \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g f^{ijk} A^j_\mu A^k_\nu\right) T^i \equiv F^i_{\mu\nu} T^i$$

$$\therefore \quad \boxed{F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g \sum_{j,k=1}^{n^2-1} f^{ijk} A^j_\mu A^k_\nu}$$

Also, $\quad F^i_{\mu\nu} = 2\,\mathrm{tr}(F_{\mu\nu} T^i)$, since $\mathrm{tr}(T^i T^j) = \frac{\delta^{ij}}{2}$

Lagrangian for $A_\mu^i(x)$ (the Yang-Mills theory):

Note that: $\mathrm{tr}\left(F'_{\mu\nu} F'^{\mu\nu}\right) = \mathrm{tr}\left(U F_{\mu\nu} \cancel{U^{-1}} \cancel{U} F^{\mu\nu} U^{-1}\right) = \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$

$\therefore \mathcal{L}_{YM} = -\frac{1}{2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$ is an SU(n) invariant generalization of Maxwell theory.

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = -\frac{1}{4} \sum_{i=1}^{n^2-1} F_{\mu\nu}^i F^{\mu\nu i}}$$

The complete gauge invariant Lagrangian:

$$\boxed{\mathcal{L} = \bar{\Psi}\left(i \not{D} - m\right) \Psi - \frac{1}{2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)}$$

SU(n) invariant gauge theory of fermions:

$$D_\mu \Psi = (\partial_\mu + igA_\mu)\Psi$$

$$F_{\mu\nu}\Psi = [D_\mu, D_\nu]\Psi$$

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

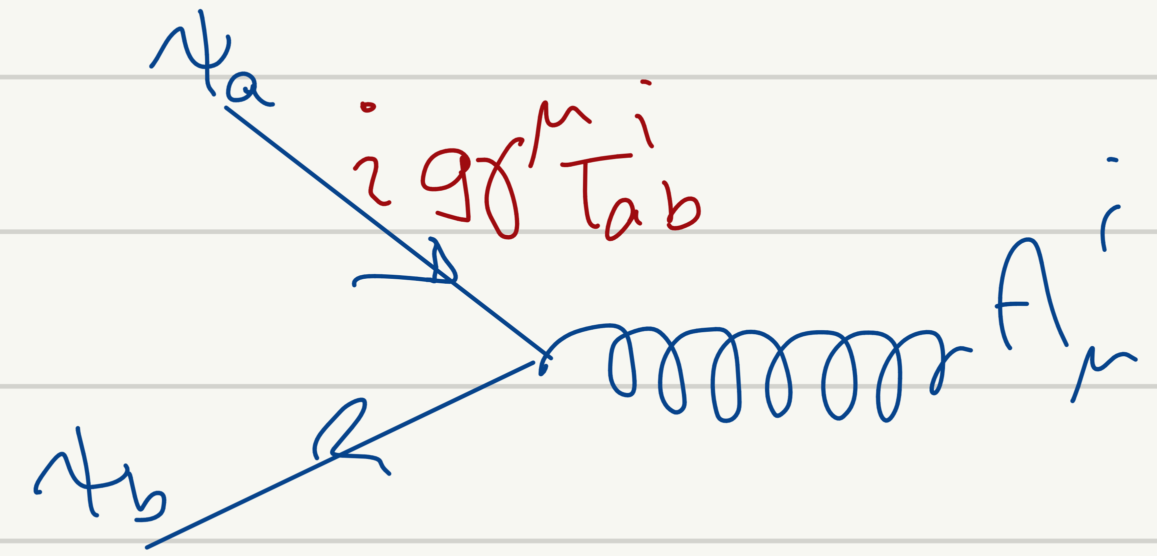
$$= \bar{\Psi}(i\partial_\mu - m)\Psi - g\underbrace{\bar{\Psi}\gamma^\mu T^i \Psi}_{J^{\mu i}} A^i_\mu - \frac{1}{4}\sum_{i=1}^{n^2-1} F^i_{\mu\nu}F^{\mu\nu i} = \mathcal{L}^0_D + \mathcal{L}_I + \mathcal{L}_{YM}$$

YM theory

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \quad \psi_a, \ (a = 1 \dots n). \qquad A_\mu = \sum_{i=1}^{n^2-1} A^i_\mu T^i, \quad F_{\mu\nu} = \sum_i F^i_{\mu\nu} T^i$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \Rightarrow \boxed{F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g f^{ijk} A^j_\mu A^k_\nu}$$

$$\mathcal{L}_D^{(0)} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \sum_{a=1}^{n}\bar{\psi}_a(i\gamma^\mu\partial_\mu - m)\psi_a$$

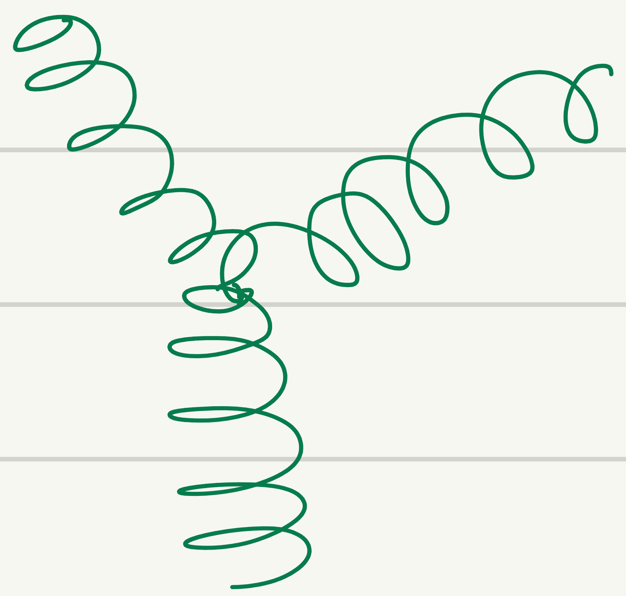


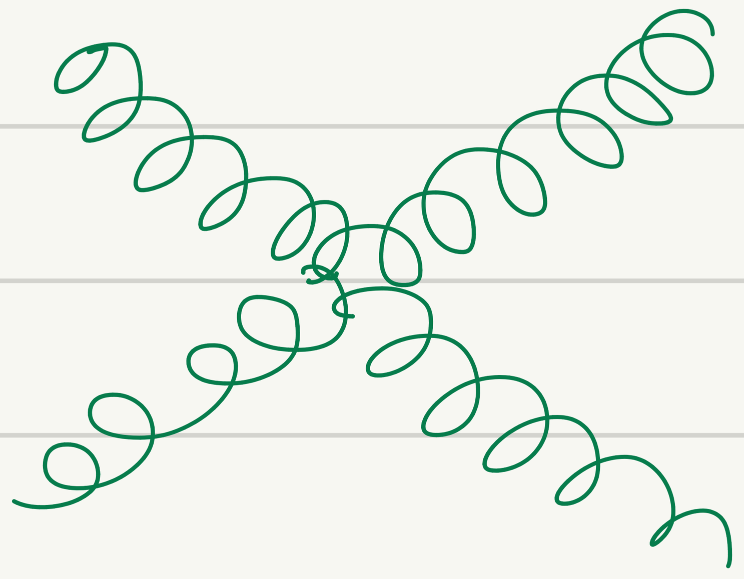
$$\mathcal{L}_I = -J^{\mu i}A^i_\mu = -g(\bar{\Psi}\gamma^\mu T^i\Psi)A^i_\mu = -g(\bar{\psi}_a\gamma^\mu(T^i)_{ab}\psi_b)A^i_\mu$$

$$\mathcal{L}_{YM} = -\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} = -\frac{1}{4}(\overbrace{\partial_\mu A^i_\nu - \partial_\nu A^i_\mu}^{F^{0i}_{\mu\nu}} - g f^{ijk}A^j_\mu A^k_\nu)(\partial^\mu A^{\nu i} - \partial^\nu A^{\mu i} - g f^{imn}A^{\mu m}A^{\nu n})$$

$$= -\frac{1}{4}F^{0i}_{\mu\nu}F^{0\mu\nu i} + g f^{ijk}\partial_\mu A^i_\nu A^{\mu j}A^{\nu k} - \frac{1}{4}g^2 f^{ijk}f^{imn}A^i_\mu A^k_\nu A^{\mu m}A^{\nu n}$$

A_μ self-interactions :





Strong Interactions

Particles with strong interactions are called hadrons. These are bound states of spin $1/2$ quarks.

$$\text{Hadrons} \begin{cases} \nearrow \text{Baryons (proton, neutron, ...): bound states of 3 quarks.} \\ \searrow \text{Mesons (pions } \pi^0, \pi^\pm, \ldots\text{): quark-antiquark bound states.} \end{cases}$$

There are 6 types of quarks q : $\boxed{\text{up}(u),\ \text{down}(d),\ \text{strange}(s),\ \text{charm}(c),\ \text{top}(t),\ \text{bottom}(b)}$

(6 quark flavours)

Quarks are Dirac particles. Notation: ψ^f, $f = u, d, s, c, t, b$.

strong interactions act on quarks. Each quark flavour has 3 colours. These transform under $SU(3)$ gauge group.

Quantum Chromodynamics (QCD): An SU(3) gauge theory

SU(3) : $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$: a "quark" with 3 "colors"

$\Psi' = U\Psi$, U: 3 x 3

But there are 6 quark "flavours" :

$\Psi^f = \Psi^u, \Psi^d, \Psi^c, \Psi^s, \Psi^t, \Psi^b$

up down charm strange top bottom

flavor index

ψ^f_c

color index

u c t : $2/3\, e$

d s b : $-1/3\, e$

$\updownarrow 1e$ + anti particles

$$\mathcal{L}^0_D = \sum_f \bar{\Psi}^f (i\not{\partial} - m_f)\Psi^f = \sum_f \sum_{a=1}^{n} \bar{\psi}^f_a (i\not{\partial} - m_f)\psi^f_a$$

To make $\mathcal{L}_D^0$ invariant under local $SU(3)$ transformations we replace $\not{\partial}\Psi^f$ by $\not{D}\Psi^f$ and add the Yang-Mills term

$$\left(\begin{array}{l} n=3 \\ n^2-1=8 \end{array} \right)$$

$$\boxed{\mathcal{L}_{QCD} = \sum_f \bar{\Psi}^f (i\not{D} - m_f)\Psi^f - \frac{1}{4}\sum_{i=1}^{8} F_{\mu\nu}^{i} F^{\mu\nu i}}$$

This is the QCD Lagrangian.

- Evidence for quarks, colour, ...
- Asymptotic freedom, confinement.