Cross-Section :

Recall :

$$\boxed{S_{fi} = \langle f|S|i\rangle = \delta_{fi} + (2\pi)^4 \delta^{(4)}\left(\sum_f p_f - \sum_i p_i\right) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathcal{M} = \delta_{fi} + F(p_i, p_f)}$$

- Transition probability: $|F(p_i, p_f)|^2$ (for $|i\rangle \to |f\rangle$)

- Transition Rate: $w = \dfrac{|F(p_i, p_f)|^2}{T}$ (T: transition time)

- Cross-Section: $\sigma(p_i, p_f) = \dfrac{w}{\text{flux}}$ (what is the incoming flux?)

- Number of final states in the range $\boxed{\vec{p}_f \text{ to } \vec{p}_f + d\vec{p}_f}$, anywhere in $V = \prod_f \dfrac{V d^3\vec{p}_f}{(2\pi)^3}$

$\therefore$

$$\boxed{d\sigma\left(p_i, p_f \to p_f + dp_f\right) = \frac{w}{\text{flux}} \prod_f \frac{V d^3\vec{p}_f}{(2\pi)^3}}$$

(see next slide)

Digression 1:

Counting the no. of states with momenta in the range $\vec{p}$ to $\vec{p}+d\vec{p}$ anywhere in volume V (for a free particle):

Method 1: ○ Minimum phase space volume occupied by 1 state:

$$(\Delta p \Delta x)^3 \geq h^3 = (2\pi\hbar)^3$$

single particle phase space: $\{x^1, x^2, x^3, p^1, p^2, p^3\}$

○ Phase space volume occupied by all states with momenta $\vec{p}$ to $\vec{p}+d\vec{p}$, anywhere in V = $V\,d^3\vec{p}$

○ Maximum no. of states in phase space volume $V\,d^3\vec{p}$

$$= \frac{V\,d^3\vec{p}}{(2\pi\hbar)^3}$$

Method 2 :

$$e^{i p_j L} = 1 \Rightarrow$$

$$p_j L = 2\pi n_j$$

$$p_j = \frac{2\pi}{L} n_j$$

- Assume that V is a cube and impose periodic boundary conditions on $e^{i\vec{p}\cdot\vec{x}}$:

$$e^{i p_j x_j} = e^{i p_j (x_j + L)} \quad \Rightarrow \quad \vec{p} = \frac{2\pi}{L}\vec{n}$$

$$\vec{n} = \{n_1, n_2, n_3\} \; : \text{ integers}$$

- For large $\vec{p}$, $\quad d\vec{p} = \frac{2\pi}{L} d\vec{n} \Rightarrow d\vec{n} = \frac{L}{2\pi} d\vec{p}$

Valid for large n.
eg:

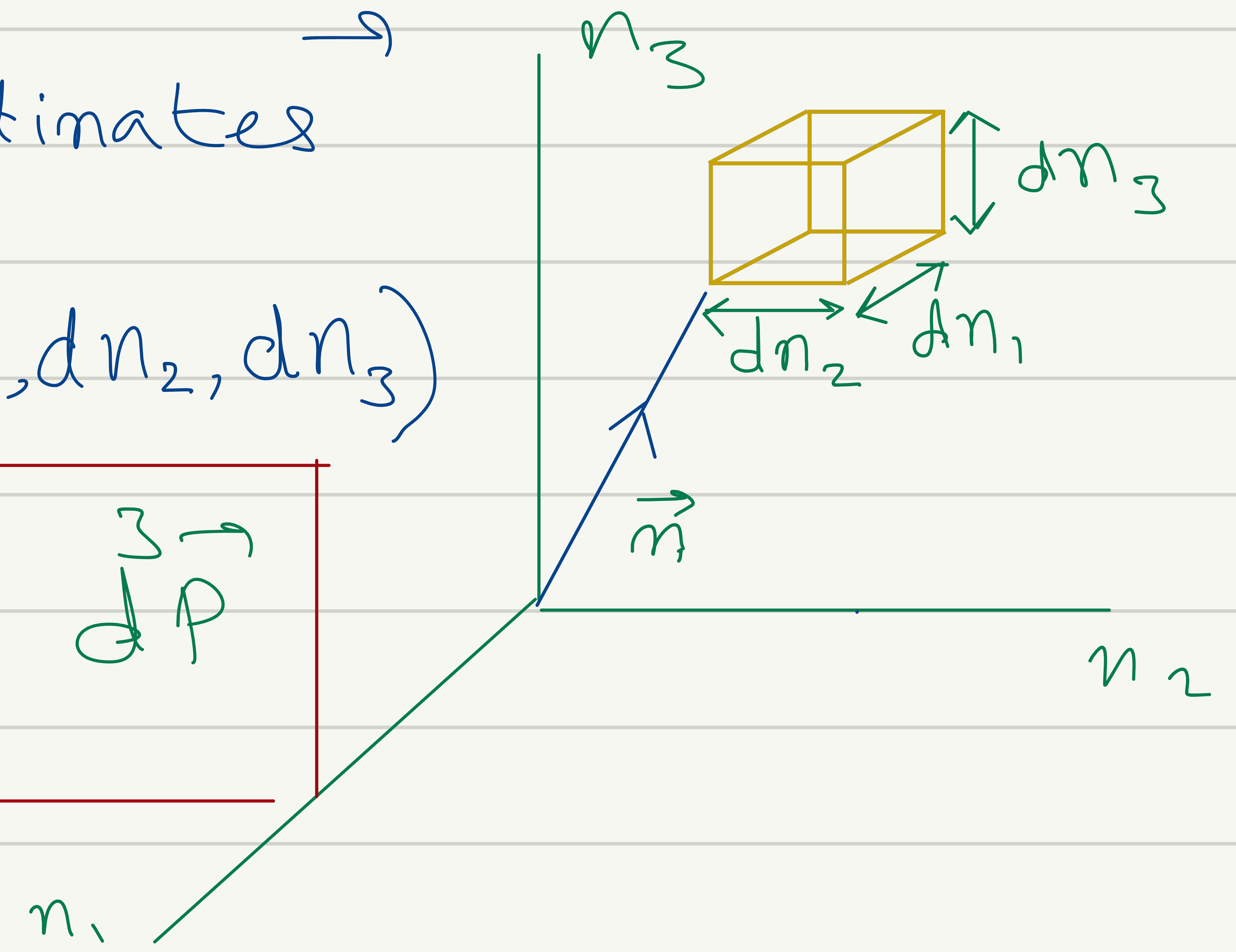
$n \sim 10^5, \; \Delta n \sim 2$

$$\frac{\Delta n}{n} \sim 0$$

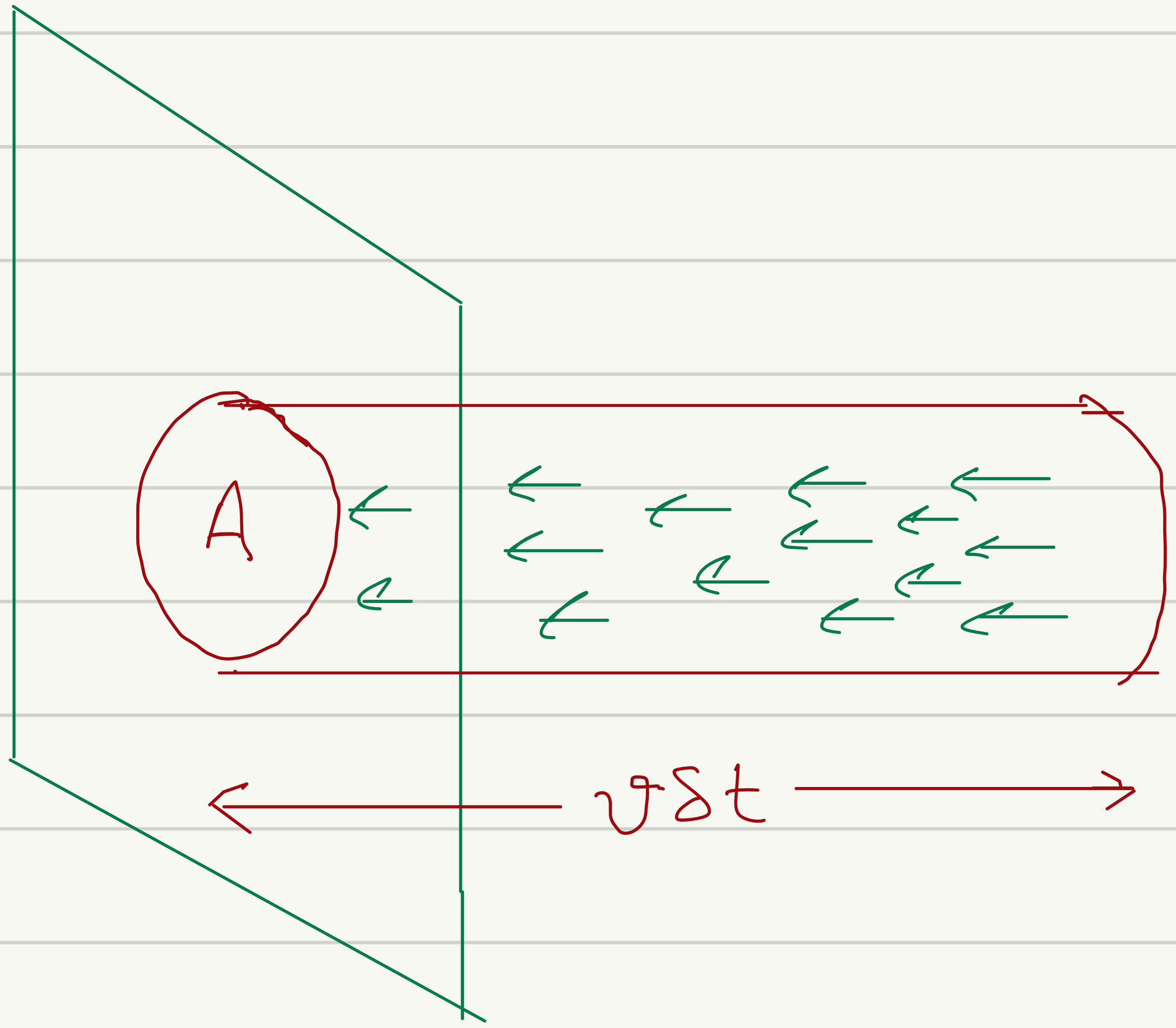
$$\Rightarrow \Delta n \sim dn$$

- No. of states = No of points with integer coordinates $\vec{\ }$ within the cube of sides (dn_1, dn_2, dn_3)

$$= \left(\begin{array}{c}\text{volume of} \\ \text{the cube}\end{array}\right) = \boxed{d^3 n = \frac{V}{(2\pi)^3} d^3 \vec{p}}$$



Digression 2 : Definition of flux (no of particles crossing a surface per unit area per unit time)



$$\boxed{\text{flux} = \frac{\rho(v\delta t)A}{A\,\delta t} = \rho v}$$

Take : $|i\rangle$ = 2 particle state :

scattering center + Scattered particle

$\Rightarrow$ stream of "1" particle incident on the target.

$v = v_{rel}$, $\rho = \dfrac{1}{V}$ $\Rightarrow$ $\boxed{\text{flux} = \dfrac{v_{rel}}{V}}$

$$\boxed{d\sigma = \frac{V\,|F(p_i, p_f)|^2}{T\, v_{rel}} \prod_f \frac{V\, d^3 p_f}{(2\pi)^3}}$$

Digression 3 :

$$F(p_i, p_f) = (2\pi)^4\, \delta^{(4)}\Big(\sum_f p_f - \sum_i p_i\Big)\, \hat{F}(p_i, p_f)$$

$$(2\pi)^4\, \delta^{(4)}(p_f - p_i) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3x\; e^{i x_\mu (p_f - p_i)^\mu}$$

$$= \lim_{\substack{V \to \infty \\ T \to \infty}} \int_{-T/2}^{T/2} dt \int_V d^3x\; e^{i x^\mu (p_f - p_i)_\mu}$$

$$\therefore\ \boxed{\Big((2\pi)^4\, \delta^{(4)}(p_f - p_i)\Big)^2 = (2\pi)^4\, \delta^{(4)}(p_f - p_i) \int_{-T/2}^{T/2} dt \int_V d^3x\, \big(e^0\big)^{=1} = TV\, (2\pi)^4\, \delta^{(4)}(p_f - p_i)}$$

Also take: $|f\rangle$: 2 - particle state of momenta

$$p_f^\mu : \quad p_1'^\mu , \; p_2'^\mu$$

Then,

$$\boxed{d\sigma = \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \frac{V^4 |F(p_1, p_2, p_1', p_2')|^2}{(2\pi)^2 \, v_{rel}} \, d^3\vec{p}_1{}' \, d^3\vec{p}_2{}'}$$

Now let's revise the main concepts we have discussed so far and then continue developing the expression for the cross section:

Cross-Section :

$$\boxed{S_{fi} = \langle f|S|i\rangle = \delta_{fi} + (2\pi)^4 \delta^{(4)}\left(\sum_f P_f - \sum_i P_i\right) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathcal{M} = \delta_{fi} + F(P_i, P_f)}$$

- Transition probability: $|F(P_i, P_f)|^2$ $(\text{for } |i\rangle \to |f\rangle)$

- Transition Rate: $w = \frac{|F(P_i, P_f)|^2}{T}$ $(T: \text{transition time})$

- Cross-Section: $\sigma(P_i, P_f) = \frac{w}{\text{flux}}$ $\quad \underline{\text{flux} = \frac{v_{rel}}{V}, \ |i\rangle = |2\text{-particles}\rangle}$

- Number of final states in the range $\vec{P}_f$ to $\vec{P}_f + d\vec{P}_f$, anywhere in $V = \prod_f \frac{V d^3\vec{P}_f}{(2\pi)^3}$

$\therefore$
$$\boxed{d\sigma(P_i, P_f \to P_f + dP_f) = \sigma(P_i, P_f) \prod_f \frac{V d^3\vec{P}_f}{(2\pi)^3}}$$

$$\boxed{d\sigma = \frac{V\,|F(p_i, p_f)|^2}{T\, v_{rel}} \prod_f \frac{V\, d^3 p_f}{(2\pi)^3}}$$

Recall: For P_μ,

$$\delta^{(4)}(P) = \delta(P_0)\,\delta(P_1)\,\delta(P_2)\,\delta(P_3) \neq 0$$

$$\Rightarrow \quad P_0 = P_1 = P_2 = P_3 = 0$$

Note: $F(p_i, p_f) = (2\pi)^4\, \delta^{(4)}\left(\sum_f p_f - \sum_i p_i\right) F'(p_i, p_f)$.

What is $(\delta(p))^2$?

$$(2\pi)^4 \delta^{(4)}(P) = \underbrace{\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3x\; e^{i x^\mu p_\mu}} = \lim_{\substack{V\to\infty \\ T\to\infty}} \int_{-T/2}^{T/2} dt \int_V d^3x\; e^{i x^\mu p_\mu}$$

$$\boxed{\left((2\pi)^4 \delta^{(4)}(P)\right)^2 = (2\pi)^4 \delta^{(4)}(P) \int_{-T/2}^{+T/2} \int_V d^3x \left(e^{i(0)}\right) = TV\,(2\pi)^4\, \delta^{(4)}(P)} \quad \text{in the limit } T, V \to \infty$$

Also assume $|f\rangle$: 2 - particle state of momenta

$$p_f^\mu : p_1'^\mu, p_2'^\mu$$

Then,

$$d\sigma = \delta^{(4)}(p_1 + p_2 - p_1' - p_2') \frac{V^4}{(2\pi)^2} \frac{|F'(p_1, p_2, p_1', p_2')|^2}{v_{rel}} d^3\vec{p}_1' \, d^3\vec{p}_2'$$

$$\boxed{d\sigma = \delta^4(p_1 + p_2 - p_1' - p_2') \frac{1}{64\pi^2} \frac{1}{(v_{rel} E_1 E_2)} \frac{\prod_\ell (2m_\ell)}{E_1' E_2'} |\mathcal{M}|^2 d^3\vec{p}_1' \, d^3\vec{p}_2'}$$

covariant expression: "$v_{rel} E_1 E_2$" $\longrightarrow \left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}$

valid in Lab frame — valid in any frame

check: $\vec{p}_2 = 0 \Rightarrow v_{rel} = v_1$

Digression 4 :

$$\text{"}E_1 E_2 v_{rel}\text{"} = \left[(P_1 P_2)^2 - m_1^2 m_2^2 \right]^{1/2}, \quad (P_1 P_2 \equiv P_{1\mu} P_2^{\mu})$$

- Lab frame :

$$\boxed{\vec{P}_2 = 0 \Rightarrow E_2 = m_2} \qquad E_1 v_{rel} = (E_1^2 - m_1^2)^{1/2} = |\vec{P}_1|$$

Recall :
$E^2 = |\vec{P}|^2 + m^2$

$$|\vec{P}_1| = m_1(v_1)\, v_1 = \frac{m_1}{\sqrt{1 - \frac{v_1^2}{c^2}}}\, v_1 \,, \qquad E_1 = m_1(v_1)\, c^2 = \frac{m_1 c^2}{\sqrt{1 - v_1^2/c^2}}$$

$$\Rightarrow \boxed{v_{rel} = |\vec{v}_1|} \quad \checkmark$$

- Center of Mass (CoM) frame :

$$\boxed{\vec{P}_1 = -\vec{P}_2} \Rightarrow P_1 P_2 = E_1 E_2 - \vec{P}_1 \cdot \vec{P}_2$$
$$= E_1 E_2 + |\vec{P}_1|^2$$

$$\Rightarrow \boxed{\text{"}E_1 E_2 v_{rel}\text{"} = |\vec{P}_1| (E_1 + E_2)}$$

Simplifications :

$$\boxed{d\sigma = \delta^4(p_1+p_2-p_1'-p_2')\frac{1}{64\pi^2}\frac{1}{\text{"}v_{rel}E_1E_2\text{"}}\frac{\Pi_\ell(2m_\ell)}{E_1'E_2'}|\mathcal{M}|^2\, d^3\vec{p}_1{}'\, d^3\vec{p}_2{}'}$$

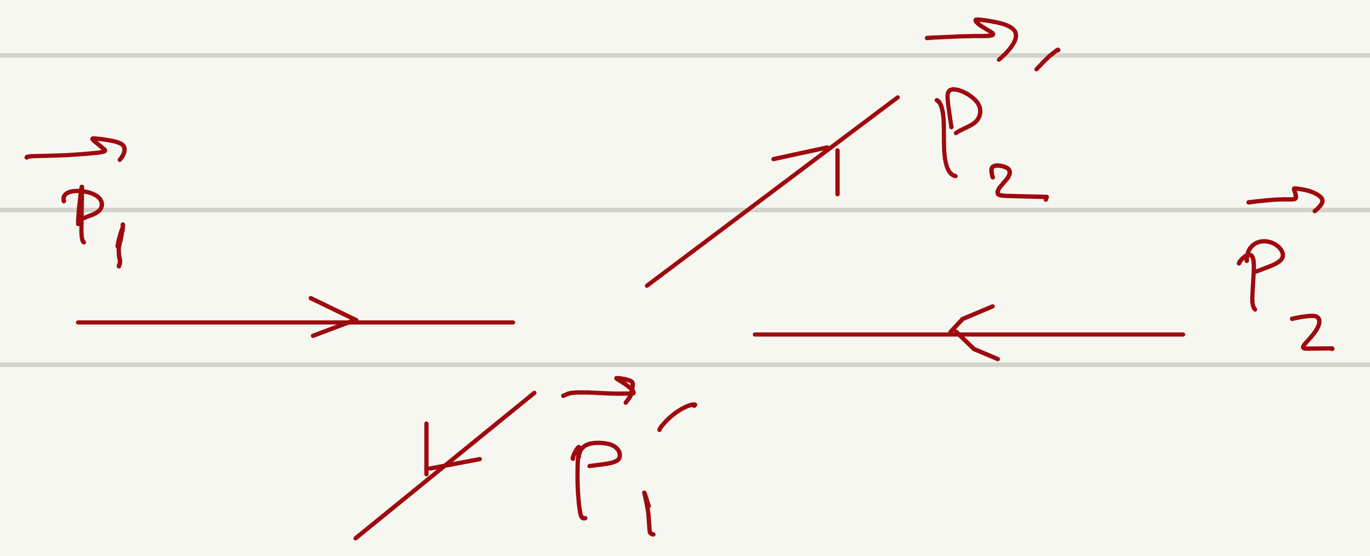
$$\delta^{(4)}\left(\sum_f p_f' - \sum_i p_i\right) = \delta(E_1+E_2-E_1'-E_2')\,\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1{}'-\vec{p}_2{}')$$

Integrate $d\sigma$ over $\vec{p}_2{}'$ $\Rightarrow$ $\underline{\vec{p}_2{}' = \vec{p}_1+\vec{p}_2-\vec{p}_1{}'}$,

$$\boxed{d\sigma = \delta(E_1+E_2-E_1'-E_2')\frac{1}{64\pi^2}\frac{1}{\text{"}v_{rel}E_1E_2\text{"}}\frac{\Pi_\ell(2m_\ell)}{(E_1'E_2')}|\mathcal{M}|^2\, d^3\vec{p}_1{}'}$$

Note: $\boxed{d^3\vec{p}_1{}' = |\vec{p}_1{}'|^2\, d|\vec{p}_1{}'|\, d\Omega_1'}$ in polar coordinates.

- $\vec{p}_2{}'$ is fully determined by $\vec{p}_1$, $\vec{p}_2$ & $\vec{p}_1{}'$
- $|\vec{p}_1{}'|$ is determined by $E_1+E_2 = E_1'+E_2'$



Determination of $|\vec{P}_1'|$:

$$E_1 = \sqrt{\vec{P}_1^{\,2} + m_1^2}, \quad E_2 = \sqrt{\vec{P}_2^{\,2} + m_2^2}, \quad E_1' = \sqrt{\vec{P}_1'^{\,2} + m_1'^2}$$

$$E_2' = \sqrt{\vec{P}_2'^{\,2} + m_2'^2} = \sqrt{|\vec{P}_1 + \vec{P}_2 - \vec{P}_1'|^2 + m_2'^2}$$

Let:

$$E_1' + E_2' - E_1 - E_2 = g(\vec{P}_1') \equiv g(|\vec{P}_1'|, \theta_{P_1'}, \phi_{P_1'})$$

Then: $\delta(E_1' + E_2' - E_1 - E_2) = \delta(g(\vec{P}_1')) \Rightarrow g(\vec{P}_1') = 0$

→ determines $|\vec{P}_1'|$ in terms of P_1^μ, P_2^μ, m_1, m_2, m_1', m_2', $\theta_{P_1'}$ and $\phi_{P_1'}$.

Now in $d\sigma = \delta(g(\vec{P}_1'))\, f(|\vec{P}_1'|)\, d|\vec{P}_1'|\, d\Omega_1'$

to implement $\delta(g)$ integrate $d\sigma$ over $|\vec{P}_1'| = x$ using:

$$\int f(x)\, \delta(g(x))\, dx = \left.\frac{f(x)}{\left(\frac{\partial g}{\partial x}\right)}\right|_{\substack{g(x)=0 \\ x = |\vec{P}_1'|}} = f(\vec{P}_1') \Big/ \frac{\partial(E_1' + E_2')}{\partial |\vec{P}_1'|}$$

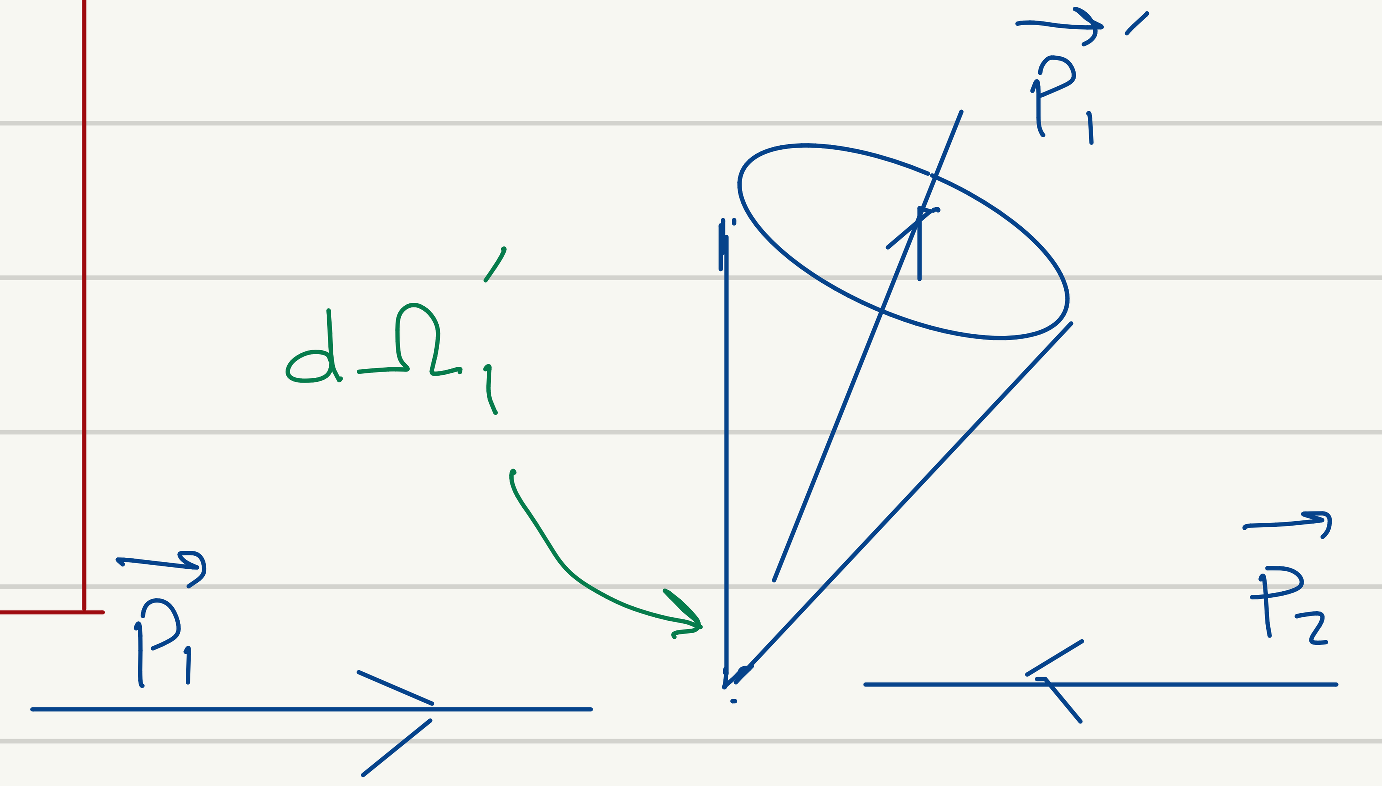
useful change of variables:

$y = g(x) \Rightarrow dy = \frac{\partial g}{\partial x} dx$

$$\Rightarrow \quad d\sigma = f(|\vec{p}_1'|, \theta_{p_1'}, \phi_{p_1'}) \left(\frac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|} \right)^{-1} d\Omega_1' \quad \left(\text{with } |\vec{p}_1'| \text{ given by } g(\vec{p}_1') = 0 \right)$$

Differential cross-section:

$$\boxed{\frac{d\sigma}{d\Omega_1'} = \frac{1}{64\pi^2} \frac{1}{\text{"}v_{rel} E_1 E_2\text{"}} \frac{\prod_\ell (2m_\ell)}{E_1' E_2'} |\vec{p}_1'|^2 \left(\frac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|} \right)^{-1} \mathcal{M}^2}$$



CoM frame:

$$\vec{p}_1 = -\vec{p}_2 \Rightarrow \vec{p}_1' = -\vec{p}_2'$$

$$E_1' = \sqrt{|\vec{p}_1'|^2 + m_1'^2} \quad , \quad E_2' = \sqrt{|\vec{p}_1'|^2 + m_2'^2}$$

$$\Rightarrow$$

$$\frac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|} = \frac{1}{2} \left(\frac{2|\vec{p}_1'|}{E_1'} + \frac{2|\vec{p}_1'|}{E_2'} \right) = |\vec{p}_1'| \frac{E_1' + E_2'}{E_1' E_2'}$$

$$\therefore \quad \boxed{\left(\frac{d\sigma}{d\Omega_1'}\right)_{CoM} = \frac{1}{64\pi^2}\frac{\Pi_\ell(2m_\ell)}{(E_1+E_2)^2}\frac{|\vec{p}_1'|}{|\vec{p}_1|}|\mathcal{M}|^2}$$

$$d\Omega_1' = d(\cos\theta_1')\, d\phi_1'$$

Total CoM cross-section:

$$\sigma^{total}_{CoM} = \int_{-1}^{+1} d(\cos\theta_1') \int_0^{2\pi} d\phi_1' \left(\frac{d\sigma}{d\Omega_1'}\right)_{CoM} \equiv \int_{4\pi} d\Omega_1' \left(\frac{d\sigma}{d\Omega_1'}\right)_{CoM}$$

For 2 identical (indistinguishable) particles in $|f\rangle$:

$$\sigma^{total}_{CoM} = \frac{1}{2}\int_{4\pi} d\Omega_1' \left(\frac{d\sigma}{d\Omega_1'}\right)_{CoM}$$

Spin & polarization information :

The amplitude $\mathcal{M}$ contains

$$u_s \,, \quad \bar{u}_s \,, \quad v_s \,, \quad \bar{v}_s \,, \quad \varepsilon^\mu_r$$

$s = 1, 2$ corresponding to spin $= \pm \frac{1}{2}$ of fermions

$r = 1, 2$ corresponding to the two polarization states of photons.

Hence $\mathcal{M} \longrightarrow \mathcal{M}_{s_1, s_2, s_3 \ldots, r_1, r_2, r_3, \ldots}$ contains information about spins and polarizations.