Covariant commutation relations (vs ETCR)

Real ϕ
$\phi^\dagger = \phi$

$$[\phi(x), \phi(y)] = ?$$

$$\phi = \phi^+ + \phi^-$$

$$[\phi(x), \phi(y)] = [(\phi^+(x) + \phi^-(x)), (\phi^+(y) + \phi^-(y))]$$

$$= \underbrace{[\phi^+(x), \phi^-(y)]} + \underbrace{[\phi^-(x), \phi^+(y)]}$$

$$= i\hbar c\, \Delta^+(x-y) + i\hbar c\, \Delta^-(x-y)$$

$$[\phi^\mp(x), \phi^\mp(y)] = 0$$

$$i\hbar c\, \Delta^-(x-y) = [\phi^-(x), \phi^+(y)] = -[\phi^+(y), \phi^-(x)] = -i\hbar c\, \Delta^+(y-x)$$

$$\Delta^-(x-y) = -\Delta^+(y-x)$$

$$[\phi^+(x), \phi^-(y)] = \sum_{\vec{k}} \sum_{\vec{k}'} \underbrace{\left(\frac{\hbar}{\omega_k}\right)^{1/2} \left(\frac{\hbar}{\omega_{k'}}\right)}\underbrace{[a_k, a^+_{k'}]}_{\delta_{\vec{k}\vec{k}'}} e^{-ikx} e^{+ik'y}$$

$$\boxed{\frac{1}{V}\sum_{\vec{k}} \to \frac{1}{(2\pi)^3}\int d^3k}$$

$$= \sum_k \left(\frac{\hbar c^2}{2V\omega_k}\right)^1 e^{-ik(x-y)}$$

$$k(x-y) = \sum_\mu k_\mu (x-y)^\mu$$

$$= \frac{\hbar c^2}{2V} \sum_{\vec{k}} \frac{1}{\omega_k} e^{-ik(x-y)}$$

$$k_0 = \frac{\omega_k}{c} = \sqrt{|\vec{k}|^2 + \mu^2}$$

$$\vec{k} = \vec{k}'$$

$$\Rightarrow \omega_k = \omega_k'$$

$$\therefore k_0 = k_0'$$

$$= \frac{\hbar c^2}{2(2\pi)^3} \int d^3k \frac{e^{-ik(x-y)}}{\omega_k} = i\hbar c \Delta^+(x-y)$$

$$\left(\Delta^+(x-y) = \frac{-ic}{2(2\pi)^3} \int \frac{d^3k}{\omega_k} e^{-ik(x-y)}\right) \to \Delta^-(x-y)$$

$$\Delta^-(x-y) = -\Delta^+(y-x) = \frac{+ic}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{+ik(x-y)}.$$

$$\Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y) = \frac{\cancel{-i}c}{(2\pi)^3}\int \frac{d^3k}{\omega_k}\, \frac{e^{ik(x-y)} - e^{-ik(x-y)}}{2i}$$

$$\boxed{\Delta(x-y) = \frac{c}{(2\pi)^3}\int \frac{d^3k}{\omega_k}\, \sin\left[k(x-y)\right]}$$

$$\boxed{[\phi(x), \phi(y)] = i\hbar c\, \Delta(x-y)}$$

we know that for $x^0 = y^0$, $\Delta(x-y) = 0$!

$\underline{\underline{\text{check}}}$:

$$\sin(k(x-y)) = \sin\big(\underbrace{k_0(x^0-y^0)}_{} + \underbrace{k_i(x^i-y^i)}_{\vec{k}\cdot(\vec{x}-\vec{y})}\big)$$

$$\sin(k(x-y))\Big|_{x^0=y^0} = \sin(\vec{k}\cdot(\vec{x}-\vec{y}))$$

$$= \sin(k|\vec{x}-\vec{y}|\cos\theta)$$

$$(y^0=0):\ \int\frac{d^3k}{\omega_k}\sin(kx)\Big|_{x^0=0} = \int\frac{d^3k}{\omega_k}\sin(\underline{\vec{k}}\cdot\vec{x}) = 0$$

$$\left[\int_{-\pi}^{\pi} d\theta\,\sin\theta = 0 \quad \sin(-\theta) = -\sin\theta\right]$$

$\underline{\underline{\text{verified}}}$

$$\boxed{[\phi(x),\phi(y)]_{x^0=y^0} = 0} \Leftarrow$$

Covariant commutation Relations

$$[\phi(x), \phi(y)] = i\hbar c\, \Delta(x-y)$$

$$\Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y)$$

$(y^\mu=0)$ $\quad \Delta^-(x) = -\Delta^+(-x)$

$$\boxed{\Delta^+(x) = \frac{-ic}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{-ikx}}$$

$$kx = k_\mu x^\mu = k_0 x^0 + k_i x^i$$

$$k_0 = \frac{\omega_k}{c} = \sqrt{|\vec{k}|^2 + \mu^2}$$

Equal times: $\boxed{[\phi(x), \phi(y)]\Big|_{x^0=y^0} = 0} \leftarrow$

Claim if $\left.[\phi(x), \phi(y)]\right|_{x^0=y^0} = 0$

then,

$[\phi(x), \phi(y)] = 0$ for all space-like x, y, $(x-y)^2 < 0$.

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1. How? $\quad (x-y)^2 = (x-y)^\mu (x-y)^\nu \eta_{\mu\nu}$

$$= \underbrace{(x^0-y^0)^2}_{\text{time}} - \underbrace{|\vec{x}-\vec{y}|^2}_{\text{space}}$$

< 0 : $\boxed{\text{space like}}$

> 0 : time like

$= 0$: light like.

if $(x-y)^2 < 0$, $\exists$ a LT $(x^\mu, y^\mu) \to (\tilde{x}^\mu, \tilde{y}^\mu)$ such that $\tilde{x}^0 - \tilde{y}^0 = 0$

converse: all $(x-y)^2 < 0$ (space like) intervals can be obtained for an equal time interval $(\tilde{x}-\tilde{y})^2 < 0$, $(\tilde{x}^0 = \tilde{y}^0)$.

$$[\phi(x), \phi(y)]\Big|_{\underset{\sim}{x^0 = y^0}} = 0 \Rightarrow \boxed{[\phi(x), \phi(y)]\Big|_{(x-y)^2 < 0} = 0}$$

2. physical implication? $\Rightarrow$ microcausality.

Two events $x^\mu=(x^0, \vec{x})$, $y^\mu=(y^0, \vec{y})$

If $(x-y)^2=(x^0-y^0)^2-|\vec{x}-\vec{y}|^2<0$

Then $\{x^\mu\}$ & $\{y^\mu\}$ are not causally connected!

$$c^2(t_x-t_y)^2<|\vec{x}-\vec{y}|^2$$

$$|\vec{x}-\vec{y}|>c(t_x-t_y)$$

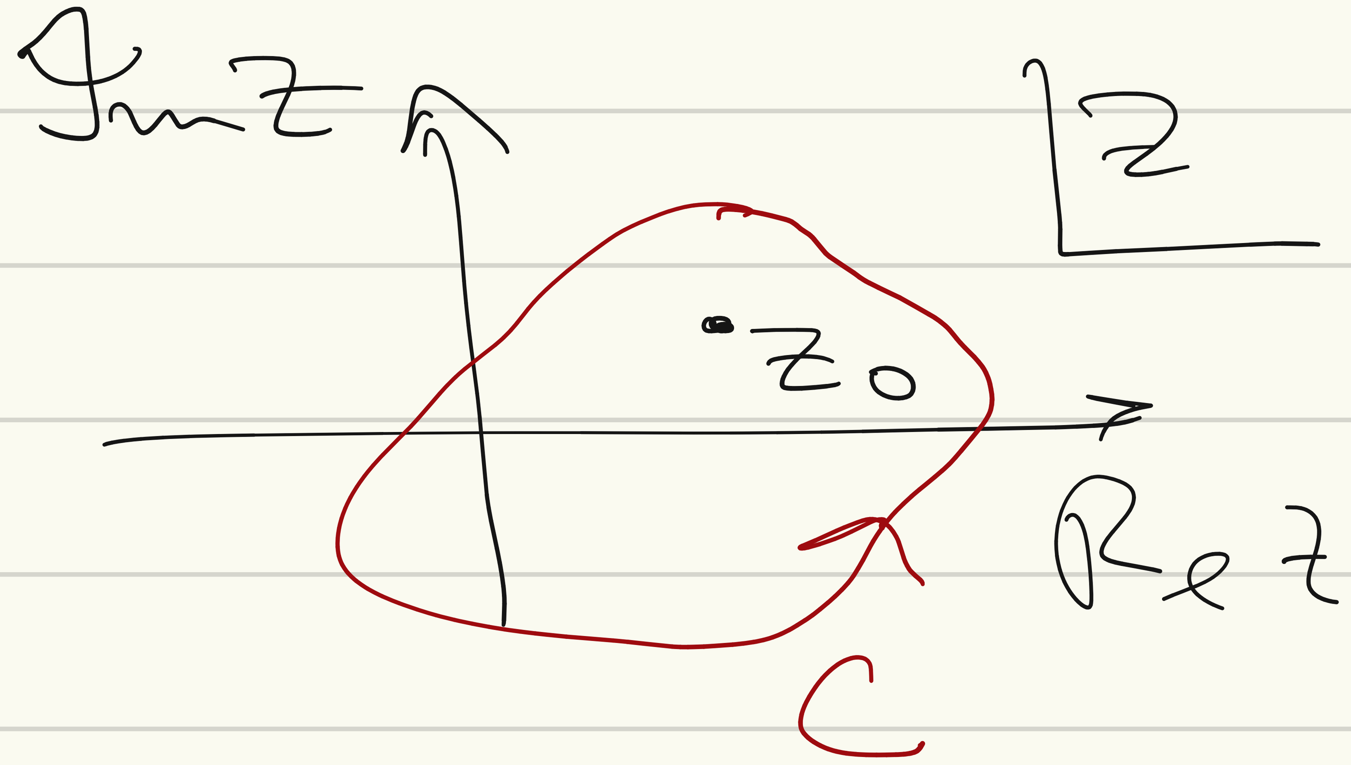
If x^μ & y^μ are causally disconnected,

then $$\boxed{\phi(x)\,\phi(y) = \phi(y)\,\phi(x)}$$ requirement of causality.

$$(x-y)^2 < 0$$

Contour Integral representation of $\Delta^{\pm}(x)$:

$$\int_C dz \left(\frac{f(z)}{z-z_0}\right) = (2\pi i)\, f(z_0)$$



$= 0$ if C does not contain z_0

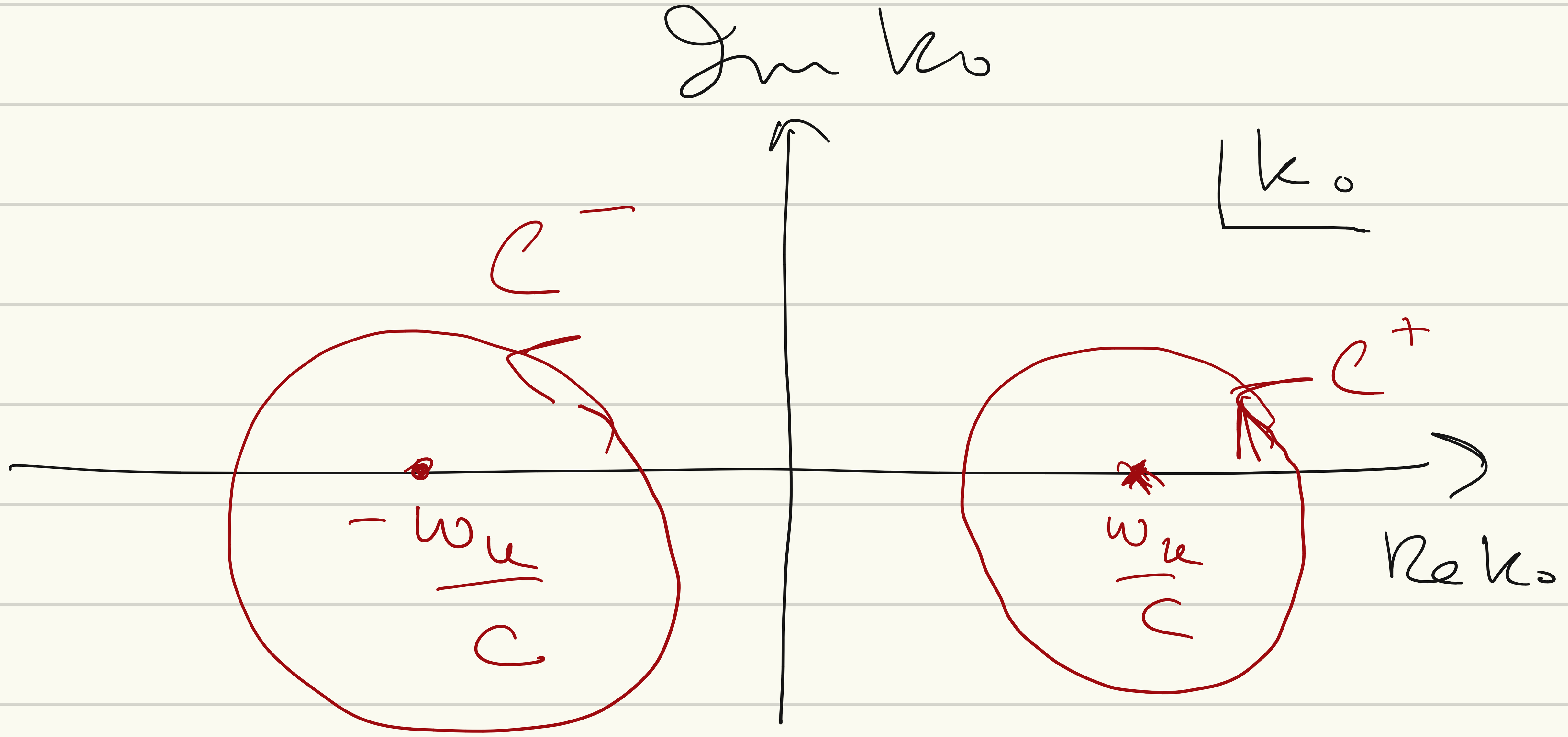
Note: $$\frac{1}{k^2-\mu^2} = \frac{1}{k_0^2 - |\vec{k}|^2 - \mu^2} = \frac{1}{\underline{\underline{k_0^2}} - (|\vec{k}|^2+\mu^2)}$$

$$\boxed{k^2 = k_\mu k^\mu = k^\mu k^\nu \eta_{\mu\nu}}$$

$$= \frac{1}{k_0^2 - \frac{\omega_k^2}{c^2}} = \frac{1}{\left(k_0+\frac{\omega_k}{c}\right)\left(k_0-\frac{\omega_k}{c}\right)}$$

$\boxed{\text{Poles at } k_0 = \pm \omega_k/c}$

Regard k_0 as complex



$$\frac{1}{k^2-\mu^2} = \frac{1}{k_0^2 - \frac{\omega_u^2}{c^2}} = \frac{1}{\left(k_0+\frac{\omega_u}{c}\right)\left(k_0-\frac{\omega_u}{c}\right)}$$

Poles at $k_0 = \frac{\omega_u}{c}$

$k_0 = -\frac{\omega_u}{c}$

Claim:

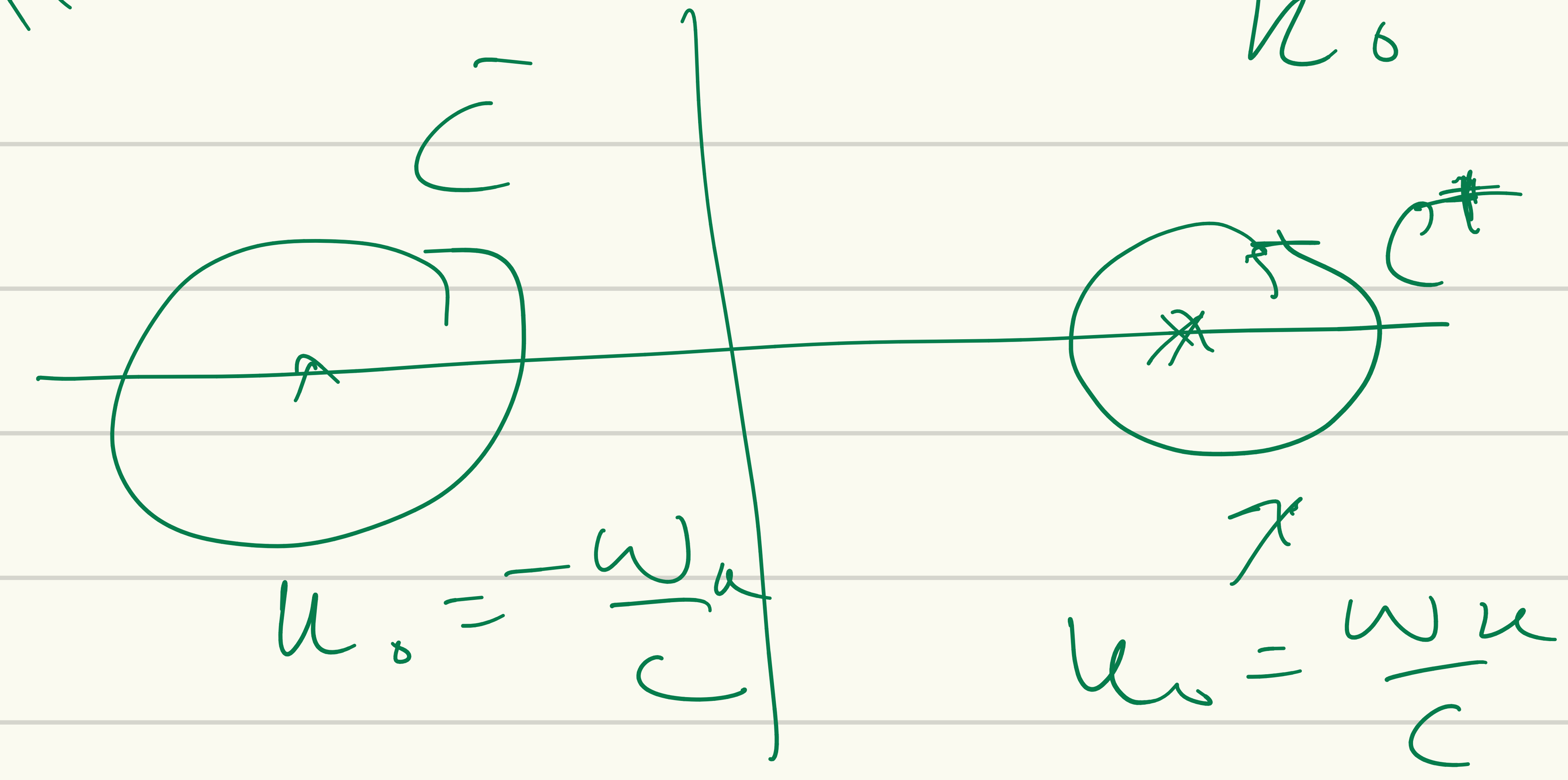
$$\Delta^{\pm} = \frac{-1}{(2\pi)^4}\int_{C^{\pm}} d^4k \, \frac{e^{-ikx}}{k^2-\mu^2}$$

$$\int_{C^{\pm}} d^4k = \int d^3\vec{k} \int_{C^{\pm}} dk_0$$

$$kx = k_0x^0 + k_i x^i$$

$$\frac{1}{k^2-\mu^2} = \frac{1}{\left(k_0+\frac{\omega_u}{c}\right)\left(k_0-\frac{\omega_u}{c}\right)}$$

$$\Delta^+(x) = \frac{-1}{(2\pi)^4}\int d^3\vec{k} \int_{C^+} dk_0 \, \frac{\overbrace{e^{-i(k_0x^0 + k_ix^i)}}^{\to f(z)}}{\left(k_0 + \frac{\omega_k}{c}\right)\underbrace{\left(k_0 - \frac{\omega_k}{c}\right)}_{1/(z-z_0)}}$$



$$= \frac{-1}{(2\pi)^4}\int d^3\vec{k}\,(2\pi i)\,\frac{e^{-i\left(\frac{\omega_k}{c}x^0 + k_ix^i\right)}}{\left(2\frac{\omega_k}{c}\right)}$$

$$= \frac{-ic}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{-ikx}$$

$kx = k_0x^0 \cancel{+} k_ix^i$

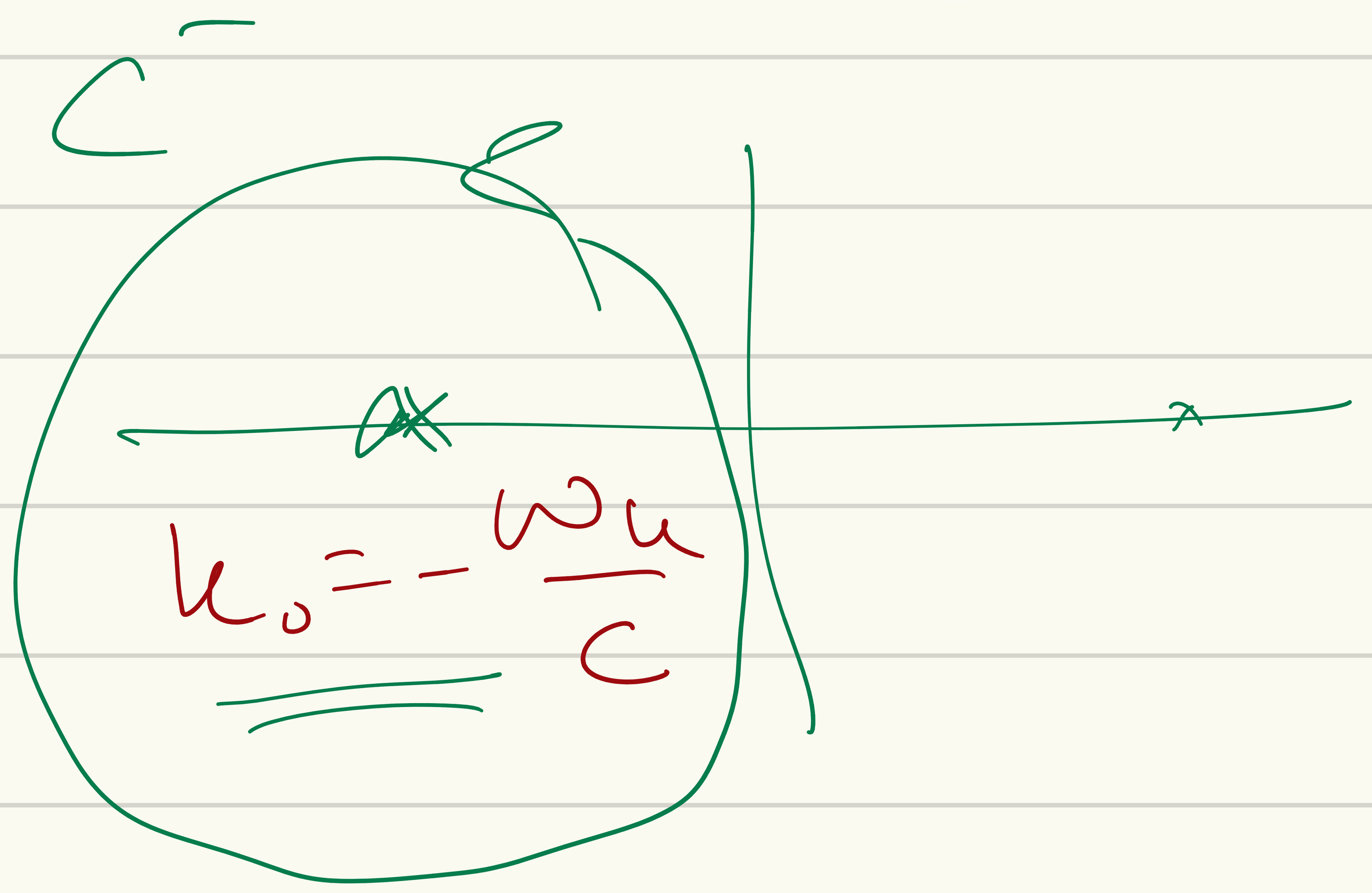
$k_0 = \frac{\omega_k}{c}$

$f(z)$

$$\Delta^-(x) = \frac{-1}{(2\pi)^4}\int d^3\vec{k}\int_{C^-} dk_0 \, \frac{e^{-i(k_0 x^0 + k_i x^i)}}{\left(k_0 + \frac{\omega_k}{c}\right)\left(k_0 - \frac{\omega_k}{c}\right)}$$

$$= \frac{-1}{(2\pi)^4}\int d^3\vec{k}\,(2\pi i)\,\frac{e^{-i\left(-\frac{\omega_k}{c}x^0 + k_i x^i\right)}}{-\left(\frac{2\omega_k}{c}\right)}$$

$(\vec{k} \to -\vec{k})$

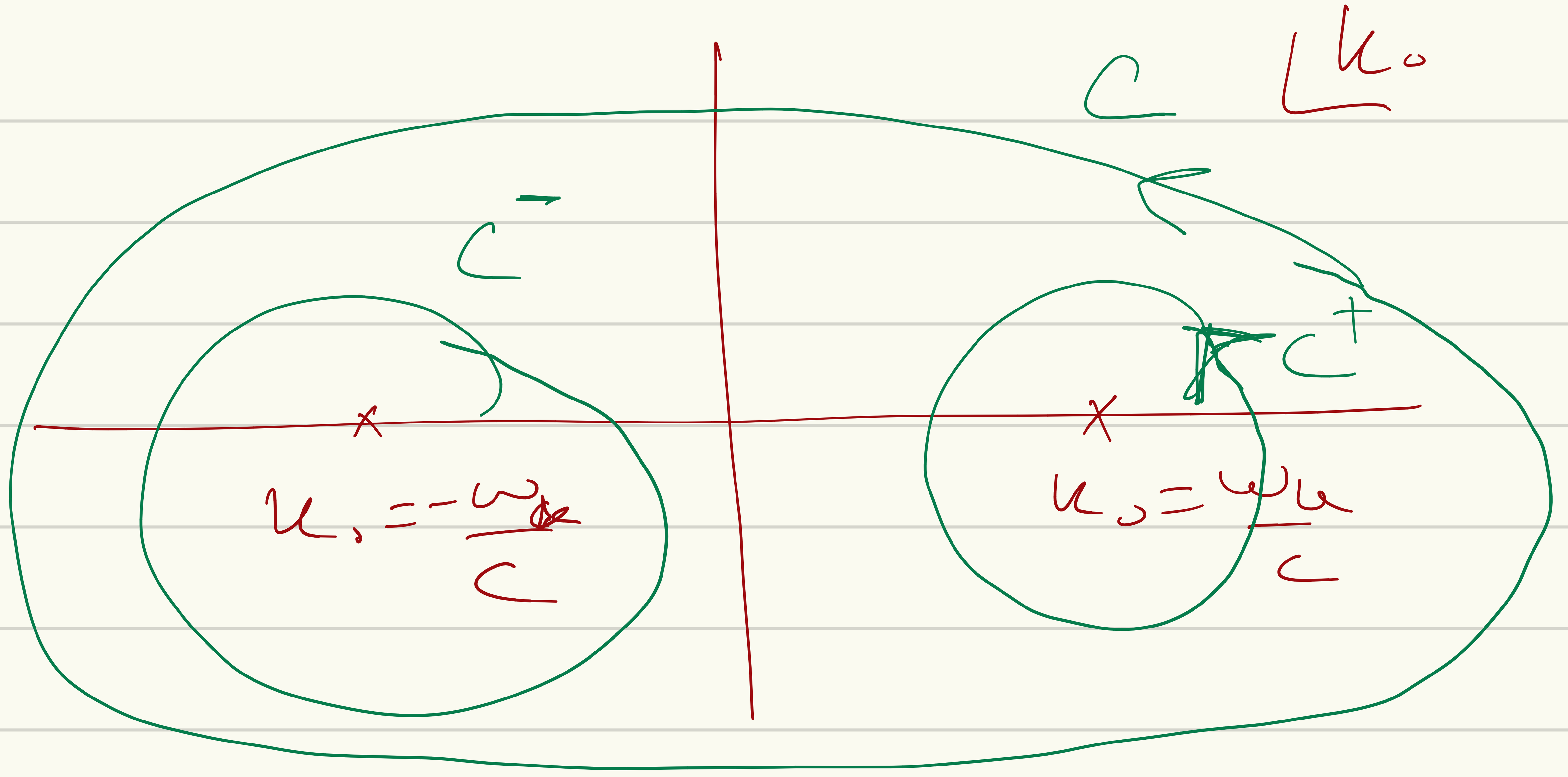


$$= \frac{ic}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{ikx} \qquad \left(kx = k_0 x^0 + k_i x^i = \left(\frac{\omega_k}{c}\right)x^0 + k_i x^i\right)$$

$$\Delta^+(x) = -\Delta^-(-x), \qquad \Delta^-(x) = -\Delta^+(-x)$$

$$[\phi(x), \phi(y)] = i\hbar c\, \Delta(x-y)$$

$$\Delta(x) = \Delta^+(x) + \Delta^-(x)$$



$$\Delta : \int_{C^+ + C^-} = \int_C$$

$$\boxed{\Delta(x) = \frac{-1}{(2\pi)^4} \int_C d^4k \, \frac{e^{-ikx}}{k^2 - \mu^2}}$$

$$\int_C dz \, \frac{f(z)}{z - z_0} = (2\pi i) f(z_0)$$

$$z \to k_0$$

Feynman propagator (Real scalar field)

$$\phi = \overbrace{\phi^+}^{a} + \overbrace{\phi^-}^{a^\dagger}$$

$$\phi^+|0\rangle = 0$$

$$(\phi^+|0\rangle)^\dagger = \langle 0|\phi^- = 0$$

consider:

$$\langle 0|\phi(x)\,\phi(x')|0\rangle$$

$$= \langle 0|\,\phi^+(x)\,\phi^-(x')|0\rangle$$

$$= \langle 0|\left(\phi^+(x)\,\phi^-(x') - \phi^-(x')\,\phi^+(x)\right)|0\rangle$$

$$= \langle 0|\,[\phi^+(x), \phi^-(x')]\,|0\rangle$$

$$= \langle 0|\,i\hbar c\,\Delta^+(x-x')\,|0\rangle$$

$$= i\hbar c\,\Delta^+(x-x')\,\langle 0|0\rangle^{=1}$$

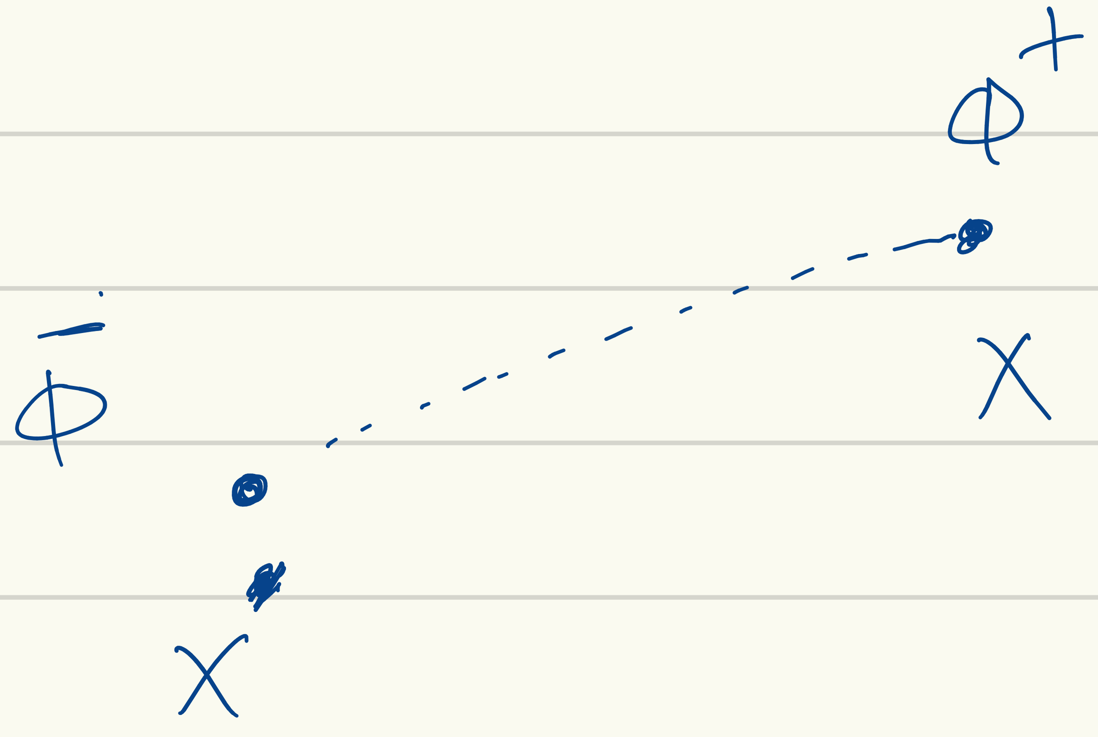
$$\boxed{\begin{aligned}&\langle 0|\phi(x)\,\phi(x')|0\rangle\\ &= i\hbar c\,\Delta^+(x-x')\end{aligned}}$$

$$x^0, \vec{x} \qquad\qquad x'^0, \vec{x}'$$

$$a_{k'} \qquad\qquad a^\dagger_{k}$$

$$\langle 0|\phi(x)\,\phi(x')|0\rangle = \langle 0|\phi^+(x)\,\bar{\phi}(x')|0\rangle = i\hbar c\,\Delta^+(x-x')$$

Physical meaning:



It makes sense

$$x'^0 < x^0$$

no sense

$$x'^0 > x^0$$

Time ordered product:

$$T\big(\phi(x)\,\phi(x')\big) = \theta(x^0-x'^0)\,\phi(x)\,\phi(x') + \theta(x'^0-x^0)\,\phi(x')\,\phi(x)$$

$$\theta(x^0) = 1 \quad x^0>0$$
$$= 0 \quad x^0<0$$

$$\langle 0 | T\left(\phi(x)\, \phi(x')\right) | 0 \rangle$$

$$= \underline{\theta(x^0 - x'^0)}\ \underline{\langle 0 | \phi(x)\, \phi(x') | 0 \rangle}$$

$$+ \underline{\theta(x'^0 - x^0)}\ \underline{\langle 0 | \phi(x')\, \phi(x) | 0 \rangle}$$

$$= i\hbar c \left(\theta(x^0 - x'^0)\, \Delta^+(x - x') + \theta(x'^0 - x^0)\, \underbrace{\Delta^+(x' - x)}_{-\Delta^-(x - x')} \right)$$

$$\boxed{\langle 0 | T\left(\phi(x)\, \phi(x')\right) | 0 \rangle = i\hbar c \left(\theta(x^0 - x'^0)\, \Delta^+(x - x') - \theta(x'^0 - x^0)\, \Delta^-(x - x') \right)}$$

The Feynman Propagator

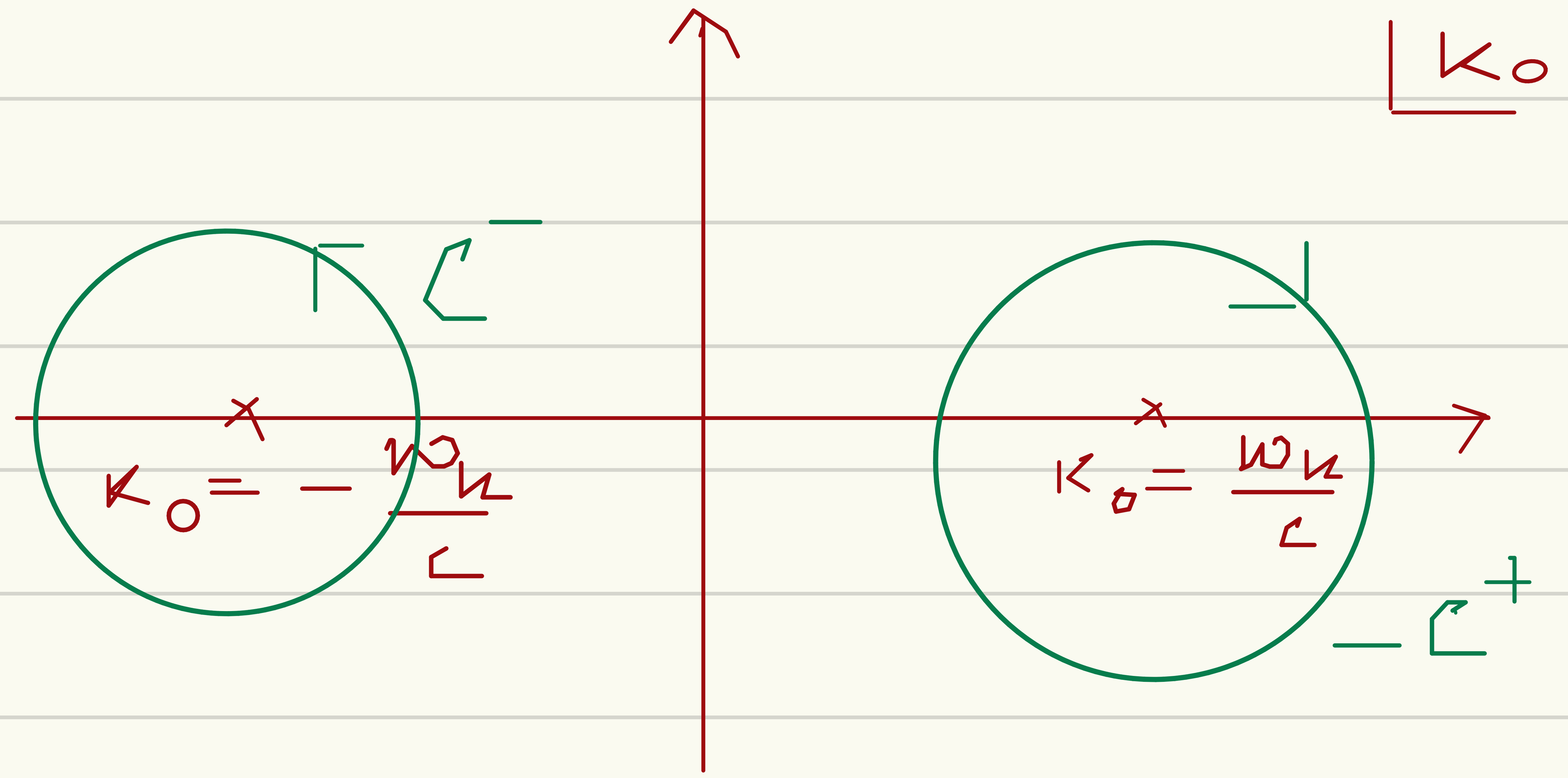
Define: $\langle 0 | T(\phi(x)\,\phi(x')) | 0 \rangle = i\hbar c\, \Delta_F(x-x')$

We have seen that: $\boxed{\Delta_F(x-x') = \theta(t-t')\,\Delta^+(x-x') - \theta(t'-t)\,\Delta^-(x-x')}$

Note that:
$$\Delta_F(x) = \Delta^+(x) \quad \text{if} \quad t > 0$$
$$= -\Delta^-(x) \quad \text{if} \quad t < 0$$

$\therefore$ We can use the contour integral representations of $\Delta^\pm$ for Δ_F

$$\Delta^+(x) = \frac{+1}{(2\pi)^4} \int_{(-C^+)} d^4k \, \frac{e^{-ikx}}{k^2 - \mu^2}$$

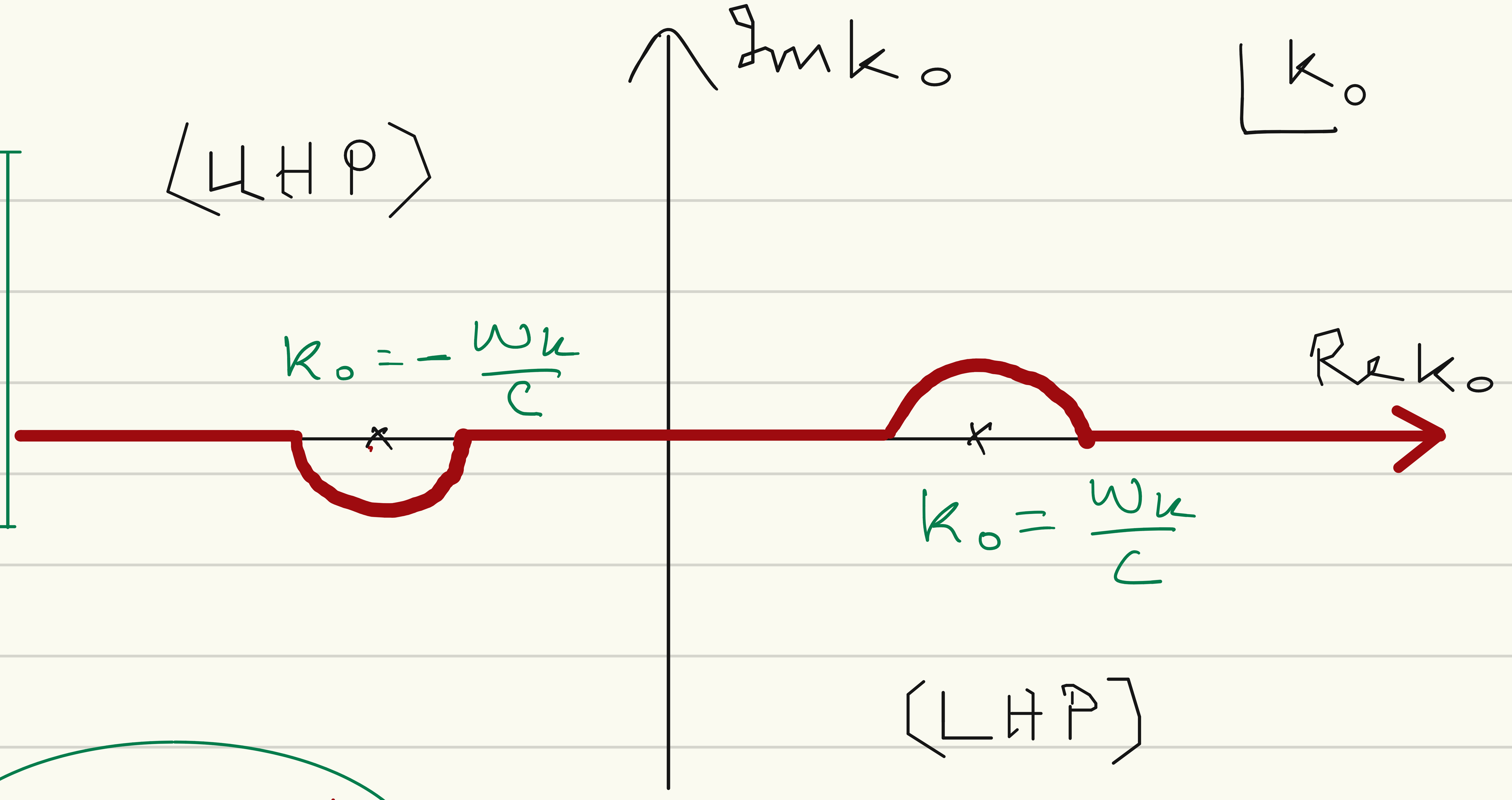


C^+: counter-clockwise, $-C^+$: clockwise

$$-\Delta^-(x) = \frac{1}{(2\pi)^4} \int_{C^-} d^4k \, \frac{e^{-ikx}}{k^2 - \mu^2}$$

The Feynman contour:

$$\boxed{\Delta_F(x) = \frac{\pm 1}{(2\pi)^4}\int_{C_F} d^4k \, \frac{e^{-ikx}}{k^2-\mu^2}}$$



close the contour in the UHP or LHP such that the extra contribution is zero, not infinite

$$e^{-ikx} = e^{-ik_0x^0}\left(e^{-ik_ix^i}\right)$$

bounded by ± 1

$$k_0 = \mathrm{Re}\, k_0 + i\, \mathrm{Im}\, k_0$$

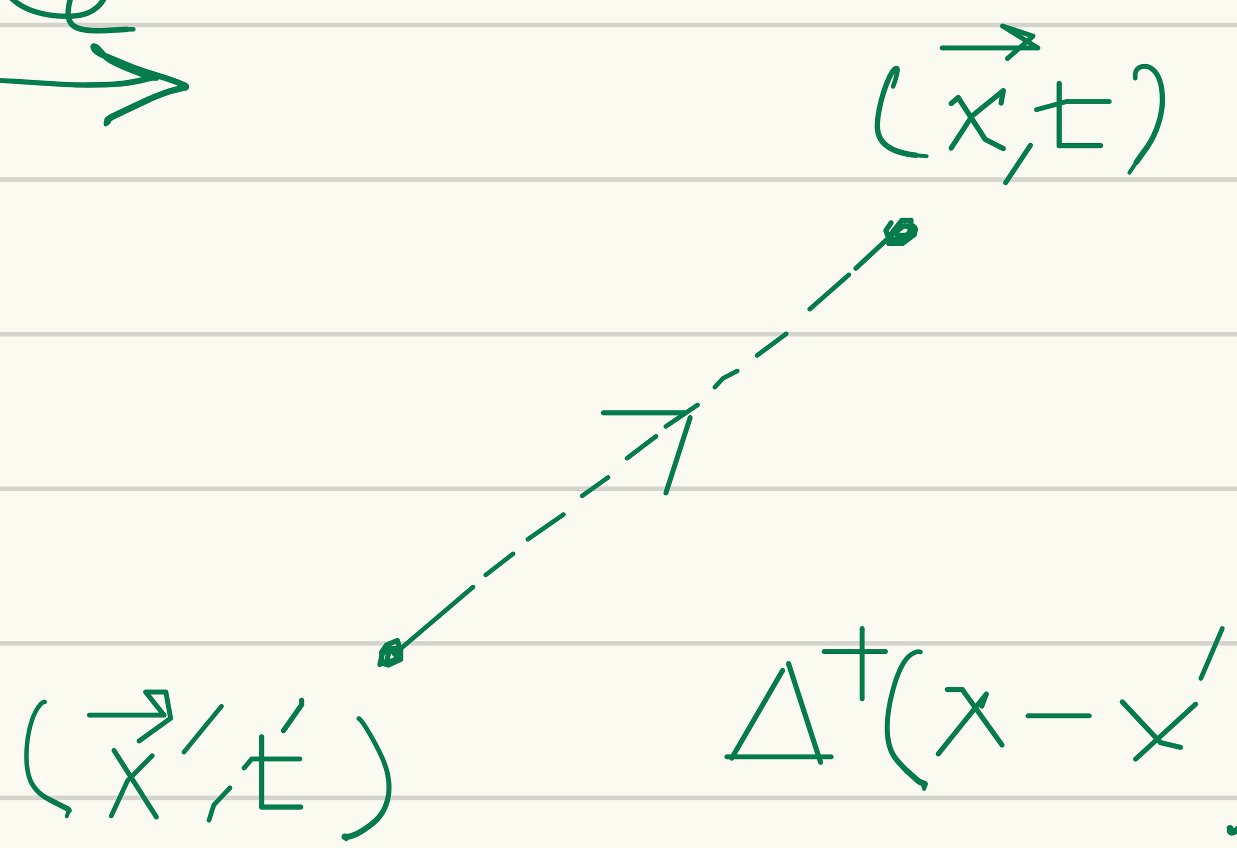
$$e^{-ik_0x^0} = e^{-i\mathrm{Re}\,k_0 x^0 - i(i\,\mathrm{Im}\,k_0)x^0} = e^{\mathrm{Im}\,k_0 x^0} \quad (*)$$

Upper half plane: $\mathrm{Im}\,k_0 \to +\infty \Rightarrow x^0 < 0 \Rightarrow \Delta_F(x) = -\Delta^-(x)$

Lower half plane: $\mathrm{Im}\,k_0 \to -\infty \Rightarrow x^0 > 0 \Rightarrow \Delta_F(x) = \Delta^+(x)$

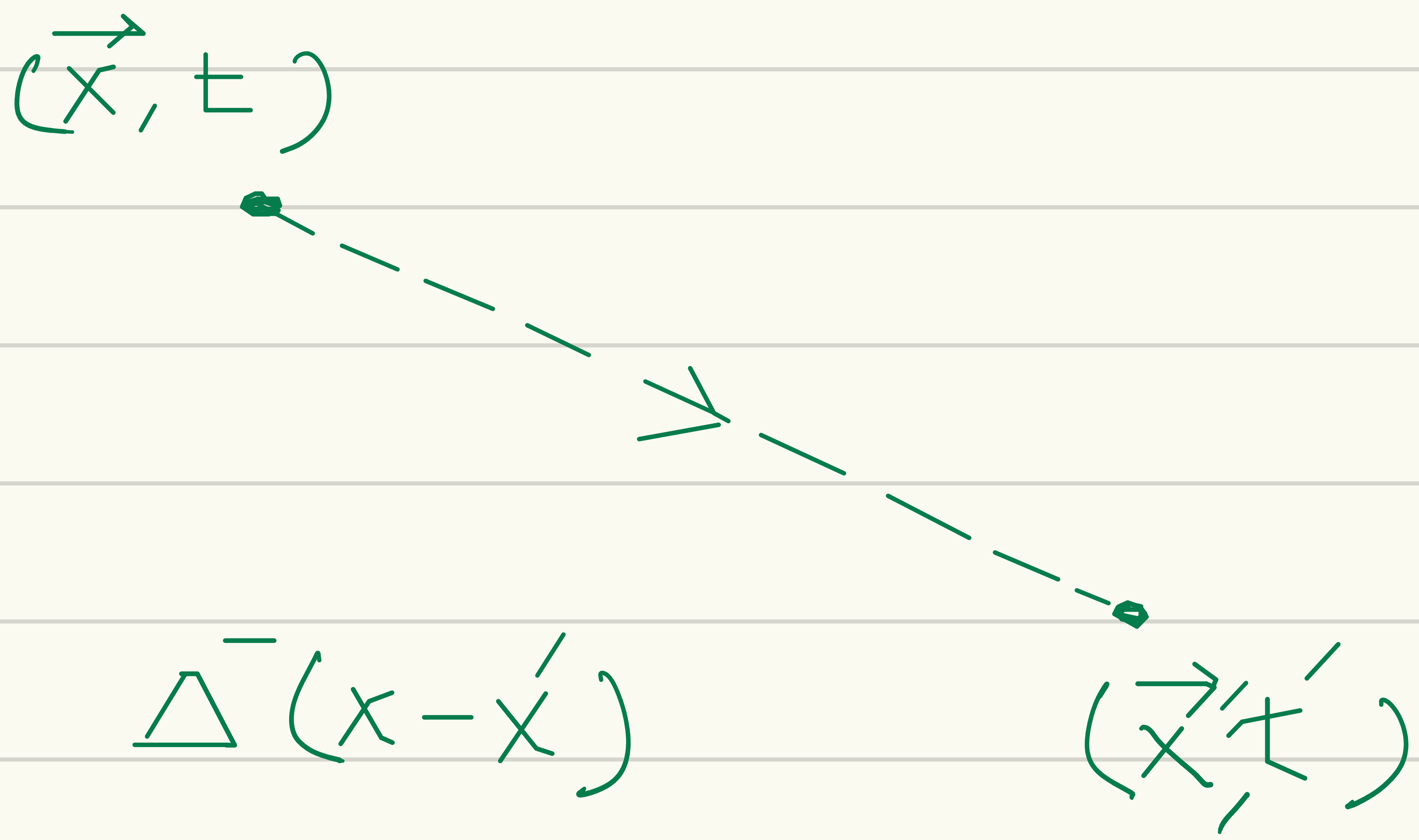
Diagramatic Representation of $\Delta_F(x-x')$

time $\longrightarrow$



$\Delta^+(x-x')$

$(t>t')$

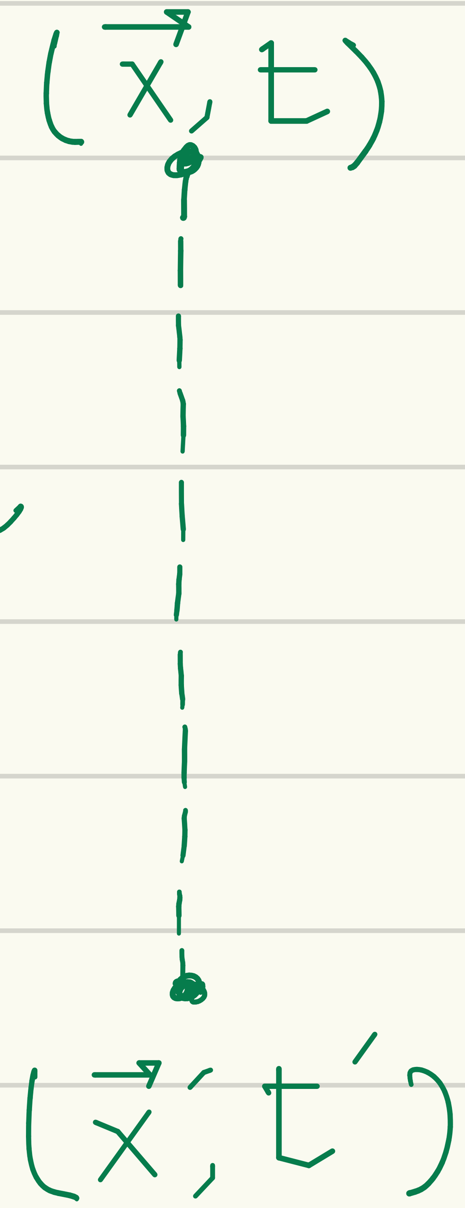


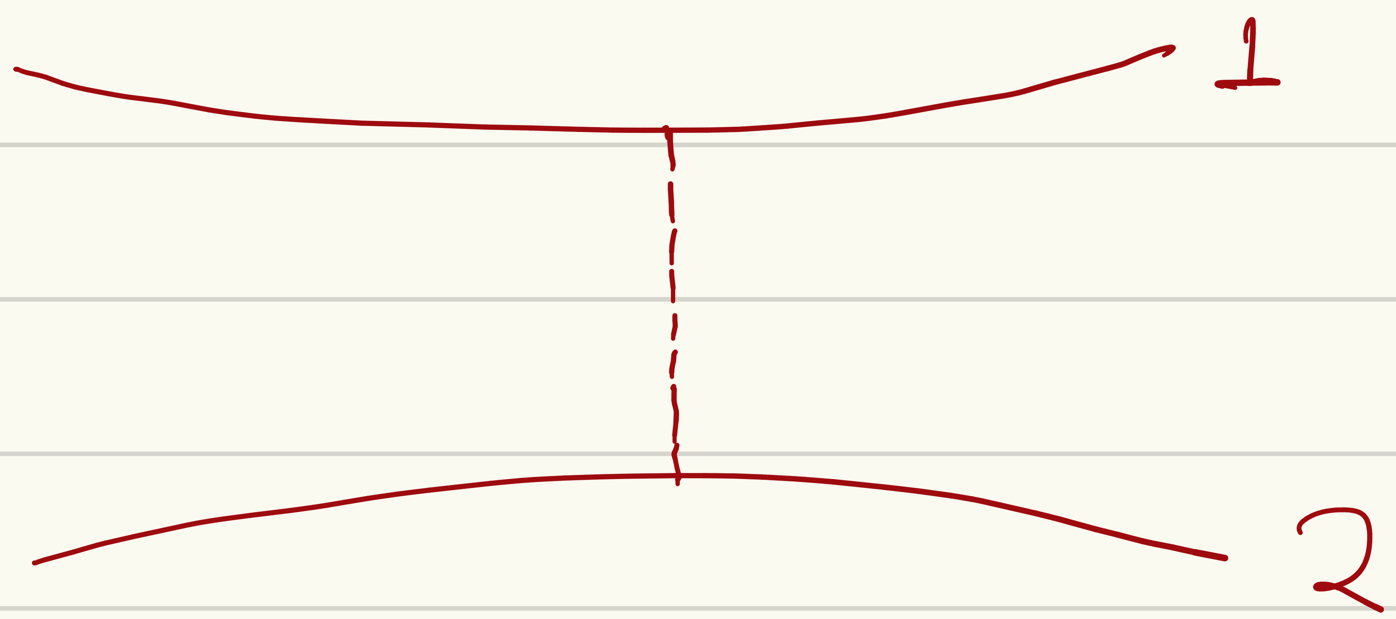
$\Delta^-(x-x')$

$(t<t')$

combined:

$\Delta_F(x-x') \sim$





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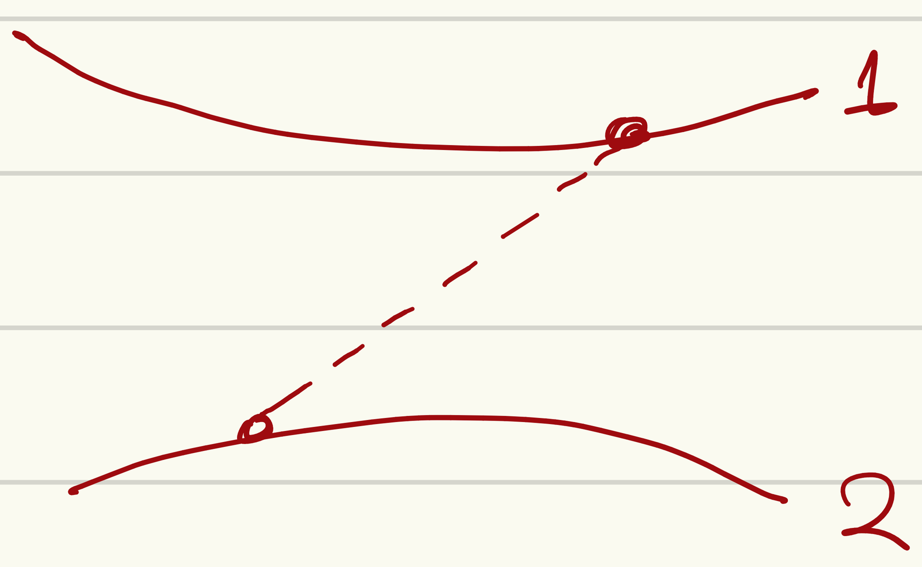

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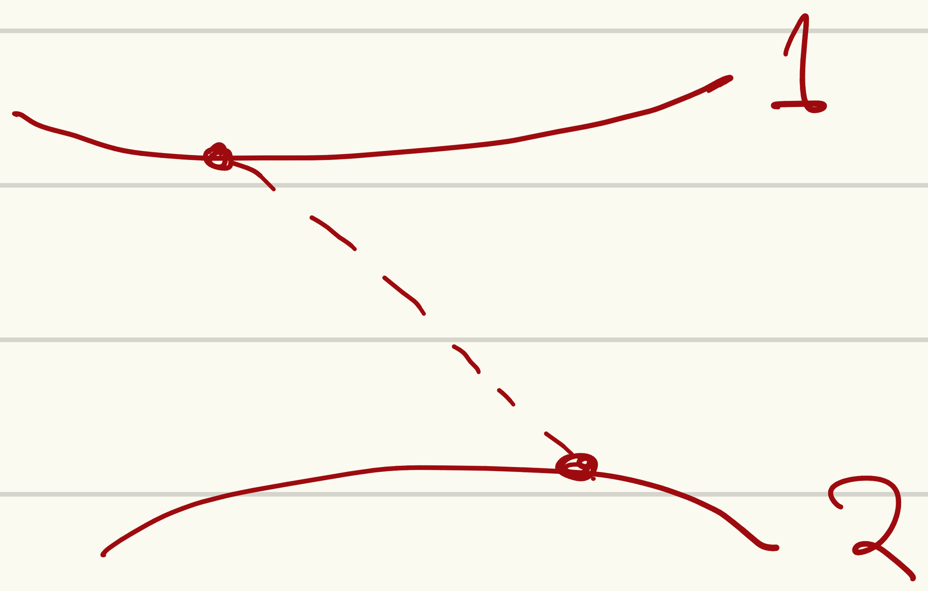
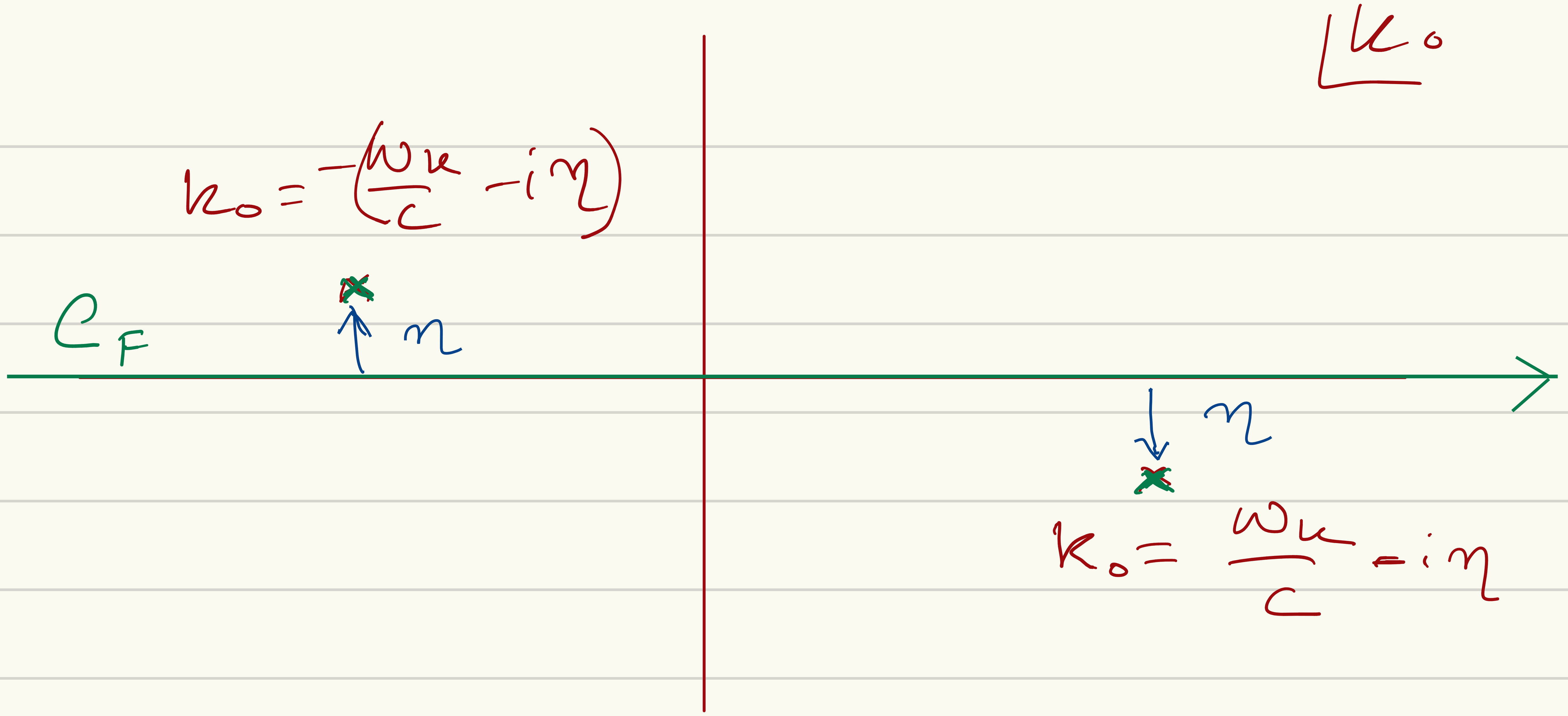


The $i\varepsilon$ prescription:



$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{C_F} d^4k \, \frac{e^{-ikx}}{k^2-\mu^2+i\varepsilon}$$

$\varepsilon > 0$, small. $(\longrightarrow 0)$

Poles: $k^2-\mu^2+i\varepsilon=0$

$$k_0^2 - \left(\frac{\omega_k^2}{c^2}\right) + i\varepsilon = 0$$

$$k_0 = \pm\sqrt{\frac{\omega_k^2}{c^2} - i\varepsilon} \approx \pm\left(\frac{\omega_k}{c} - i\eta\right)$$

$$\boxed{\epsilon = 2\eta\,\omega_k/c}$$

$$\left(k_0 - \left(\frac{\omega_k}{c} - i\eta\right)\right)\left(k_0 + \left(\frac{\omega_k}{c} - i\eta\right)\right)$$

$$= k_0^2 - \frac{\omega_k^2}{c^2} + i\,\overset{=\epsilon}{\boxed{\frac{2\eta\,\omega_k}{c}}} + \cancel{\eta^2} \quad (\text{for } \eta\to 0)$$

Propagators and Greens functions:

Consider: $$(\Box_x + \mu^2)\phi(x) = J(x)$$

The Greens function for this equation is given by

$$(\Box_x + \mu^2)\, G(x-y) = -\delta^4(x-y)$$

Then ϕ soln is then,

$$\phi(x) = \phi_0(x) - \int d^4y\, G(x-y)\, J(y)\,,$$ where, $(\Box - \mu^2)\phi_0 = 0.$

<u>Check</u>:

$$(\Box_x + \mu^2)\phi(x) = -\int d^4y\, (\Box_x + \mu^2)\, G(x-y) J(y) = \int d^4y\, \delta^4(x-y)\, J(y)$$

$$= J(x) \quad \checkmark$$

Solution for $G(x-y)$

In Fourier space,

$$G(x-y) = \int d^4k \, \tilde{G}(k) \, e^{-ik(x-y)}$$

$$\Box_x G(x-y) = \int d^4k \, (-k_\mu k^\mu) \, \tilde{G}(k) \, e^{-i(k-y)}$$

$$\delta^4(x-y) = \frac{1}{(2\pi)^4} \int d^4k \, e^{-ik(x-y)}$$

Hence:

$$(k^2-\mu^2) \, \tilde{G}(k) = \frac{1}{(2\pi)^4} \Rightarrow \boxed{\tilde{G}(k) = \frac{1}{(2\pi)^4} \frac{1}{k^2-\mu^2}}$$

$$\boxed{G(x-y) = \frac{1}{(2\pi)^4} \int_C d^4k \, \frac{e^{-ik(x-y)}}{k^2-\mu^2}}$$

The integrand has poles at $k^2 = \mu^2 \Rightarrow k_0 = \pm \omega_k/c$, so the integral is not well defined until we specify how to handle the poles. This is done by promoting k_0 to a complex variable and specifying a contour like C^+, C^- or C_F. The contour choice leads to different types of Greens functions.

Hence $\Delta_F(x-y)$ is the Greens function evaluated for the Feynman contour.

$$\Delta_F(x-y) = G(x-y)\Big|_{C_F}$$

In QFT, $\Delta_F(x-y)$ is the Feynman propagator in the free (non-interacting) theory. The expression for $\langle 0|T(\phi(x)\phi(y))|0\rangle$ changes in the presence of interactions (as we will see).

The correspondence with the Greens function holds in the free theory although the terminology is retained in general: $\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle$: n-point Greens function.