MN3 Institutionen för teoretisk fysik
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Examination in Quantum Mechanics

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To be used: Mathematics Handbook, Physics Handbook and pocket calculator.

5 problems in 6 hours.
Each problem gives a maximum of 5 points.
Read the problems through before you start solving them.
Please start each solution on a new paper.

1. a) The eigenfunctions $|\psi_n\rangle$ of an operator $\hat{A}$ ($\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$) form an orthonormal discrete basis. Show that the matrix of the operator $\hat{A}$ is diagonal in such a basis.

b) If $\hat{B}$ is another operator such that $[\hat{A}, \hat{B}] = 0$, show that $\hat{B}$ is also diagonal in the same basis (assume the spectrum of $\hat{A}$ is not degenerate).

c) Show that if a unitary operator $\hat{U}$ can be written in the form $\hat{U} = \mathbf{I} + i\epsilon\hat{F}$, where ϵ is a real infinitesimally small number, the operator $\hat{F}$ is Hermitian.

2. a) Use the time evolution equation in the Heisenberg picture,

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{A},\ \hat{H}\right] ,$$

to show that the operator equations of motion for a free particle are:

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}}{m} .$$

b) How are such equations modified for an harmonic oscillator having the Hamiltonian $\hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2$?

c) Show that the oscillator has the minimal uncertainty $\Delta x \Delta p$ when it is in its ground state $|0\rangle$.

d) How much bigger is such uncertainty if the oscillator is in the first exited state $|1\rangle$?

Hints: the $|n\rangle$ states are the eigenfunctions of the operator $N = a^\dagger a$. Use that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

3. A particle with mass m go through the potential

$$V(x) = \begin{cases} 0 & , \quad x < 0 \\ V_0 & , \quad 0 < x < a \\ 0 & , \quad x > a. \end{cases}$$

with $V_0 > 0$ (assume the particle is coming form $x = -\infty$).

a) Give the general definition of the reflection (R) and transmission (T) coefficients.

b) Assuming $E = \frac{3}{2}V_0$, determine the expression giving the smallest value of a such that $T = 1$ and describe how this depend on the energy of the particle.

c) Discuss, shortly, the physical interpretation of this "total transmission" effect and give the relation between the barrier thickness a and the DeBroglie wavelength of the particle in those cases in which it is realized.

d) What are the asimtotic values for T and R in the limit $m \to \infty$?

4. A particle with mass m (spin is neglected) in a central potential $V(r)$ is described by the time independent wave function

$$\psi(x,y,z) = \sqrt{\frac{15}{8\pi}} \frac{1}{(a)^{3/2}} \frac{xz}{r^2} e^{-r/2a}$$

where a is a constant having the same dimension of r, and $r^2 = x^2 + y^2 + z^2$.

a) Show that the wave function is normalized.

b) Verify that the only possible value of the orbital angular momentum quantum number is $l = 2$ and determine the probability of finding the values $\hbar$ and $-\hbar$ when measuring the z-component of the orbital angular moment of the particle.

c) Determine the probability of finding the particle at a distance $r > 2a$ from the origin.

d) If the potential is given by $V(r) = -\frac{\hbar^2}{2mra}$ determine what is the value of the energy E.

Hint: $\int_0^\infty r^k e^{-r/a}\, dr = k!a^{k+1}$.

5. Consider the spin state of a spin-$\frac{1}{2}$ particle

$$\chi = \begin{pmatrix} \frac{1}{2} \\ -i\sqrt{\frac{3}{4}} \end{pmatrix}$$

in the representation where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

a) What is the probability of finding the z-component of the spin being $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$?

b) Determine the matrix elements of the operator S_x^2 in such representation and the expectation value of S_x^2 on the χ state.

c) What would be the dimension of the matrices of the $\mathbf{S}^2$ and $S_{x,y,z}$ operators if the particle had instead spin 1 ?

GOOD LUCK!

Solutions will be posted at Institutionen för teoretisk fysik.