F3EPT
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Solutions to Examination in Quantum Mechanics

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Also available on `http://www.teorfys.uu.se/~edsjo/kmek/index.html`

1. a) Let ψ_n be the common eigenfunctions of $\hat{A}$ and $\hat{B}$ with eigenvalues a_n and b_n respectively. Then consider the expression

$$\hat{A}\hat{B}\psi_n = \hat{A}b_n\psi_n = b_n\hat{A}\psi_n = b_n a_n\psi_n = a_n b_n\psi_n = a_n\hat{B}\psi_n = \hat{B}a_n\psi_n = \hat{B}\hat{A}\psi_n$$

i.e. we have

$$(\hat{A}\hat{B} - \hat{B}\hat{A})\psi_n = [\hat{A},\hat{B}]\psi_n = 0.$$

Since we can expand an arbitrary wave function Ψ in the eigenfunctions ψ_n, $\Psi = \sum_n c_n\psi_n$ we have

$$[\hat{A},\hat{B}]\Psi = \sum_n c_n[\hat{A},\hat{B}]\psi_n = 0$$

which means that $[\hat{A},\hat{B}] = 0$ since Ψ is arbitrary.

b) Let ψ_n be the eigenfunctions of $\hat{A}$, i e $\hat{A}\psi_n = a_n\psi_n$. Now consider

$$\hat{A}\hat{B}\psi_n = \left\{[\hat{A},\hat{B}] = 0\right\} = \hat{B}\hat{A}\psi_n = \hat{B}a_n\psi_n = a_n\hat{B}\psi_n.$$

This means that $\hat{B}\psi_n$ is an eigenfunction of $\hat{A}$ with eigenvalue a_n. Since the eigenvalue a_n is non-degenerate and eigenfunctions corresponding to different eigenvalues are linearly independent $\hat{B}\psi_n$ can only differ from ψ_n by a constant factor, i e

$$\hat{B}\psi_n = b_n\psi_n$$

which means that ψ_n are simultaneous eigenfunctions of $\hat{A}$ and $\hat{B}$.

c) Consider the expectation value of an Hermitian operator $\hat{A}$ with respect to an eigenfunction ψ_n

$$\langle A\rangle = \langle\psi_n|A|\psi_n\rangle = \langle\psi_n|a_n\psi_n\rangle = a_n\langle\psi_n|\psi_n\rangle = a_n \tag{1}$$

but since $\hat{A}$ is Hermitian we can as well write this as

$$\langle A\rangle = \langle\psi_n|A|\psi_n\rangle = \langle A\psi_n|\psi_n\rangle = \langle a_n\psi_n|\psi_n\rangle = a_n^*\langle\psi_n|\psi_n\rangle = a_n^*. \tag{2}$$

Since Eqs. (1) and (2) are equal we get

$$a_n = a_n^*$$

which means that a_n is real.

2. a) i) Potential 1) is invariant under parity.

ii) The eigenfunctions to potential 1) will have definite parity (odd or even) and the eigenfunctions of potential 2) and 3) will not have any definite parity.

iii) Potential 1) will have discrete energy spectrum when $E < V_0$ (if V_0 is big enough) and continous energy spectrum when $E > V_0$. Potential 2) will only have discrete energy spectrum. Potential 3) will have discrete energy spectrum when $E < V_0$ (if V_0 is big enough) and continous energy spectrum when $E > V_0$.

b) i) Potential 1) and 4) are separable in spherical coordinates.
   ii) Potential 1), 2) and 3) are separable in cartesian coordinates.
   iii) All four potentials are invariant under parity.

c) i) All three wave functions are suitable for two non-identical particles.
   ii) Since the wave function for two identical fermions has to be antisymmetric with respect to exchange of particle 1 and 2 only wave function number 3) will do.
   iii) Since the wave function for two identical bosons has to be symmetric with respect to exchange of particle 1 and 2 only wave function number 1) will do.

d) i) For each n, l can take on values from 0 to $n-1$ and for each l, m can take on the $2l-1$ values from $-l$ to l. Hence the degeneracy of a state E_n is given by
$$\sum_{l=0}^{n-1}(2l+1) = n^2.$$
The state with $n=3$ thus has the degeneracy 9.
   ii) That the z component of the orbital angular momentum is $2\hbar$ means that $m=2$. The lowest possible value of l is then $l=2$ which means that the lowest possible value of n is $n=l+1=3$.
   iii) The closest higher energy state is E_4 and hence the required energy is
$$\Delta E = E_4 - E_3 = -\frac{\hbar^2}{2\mu a_0^2}\left[\frac{1}{4^2}-\frac{1}{3^2}\right] = \frac{7\hbar^2}{288\mu a_0^2}$$

3. The transmission and reflection coefficients T and R are defined as

$$T = \frac{|S_{\text{trans}}|}{|S_{\text{inc}}|} \tag{3}$$
$$R = \frac{|S_{\text{refl}}|}{|S_{\text{inc}}|} \tag{4}$$

where S_{inc}, S_{trans} and S_{refl} are the incoming, transmitted and reflected probability currents respectively.

The time independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(x)u(x) = Eu(x)$$

yields

$$\begin{cases} \frac{d^2u}{dx^2} + \frac{2mEu(x)}{\hbar^2} = 0 & , \quad x<0 \\ \frac{d^2u}{dx^2} + \frac{2m(E-V_0)u(x)}{\hbar^2} = 0 & , \quad x>0. \end{cases}$$

For convenience, now introduce

$$\begin{cases} k = \sqrt{\frac{2mE}{\hbar^2}} \\ K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \end{cases}$$

which gives us the solution

$$u(x)\begin{cases} Ae^{ikx} + Be^{-ikx} & , \quad x<0 \\ Ce^{iKx} + De^{-iKx} & , \quad x>0. \end{cases}$$

Since we cannot have any reflected wave in the region $x>0$ we must have $D=0$. Continuity of u and $\frac{du}{dx}$ at $x=0$ yields

$$\begin{cases} A+B=C \\ k(A-B)=KC \end{cases}$$

which has the solution

$$\begin{cases} B = \frac{k-K}{K+k} A \\ C = \frac{2k}{K+k} A. \end{cases}$$

The incoming, transmitted and reflected probability currents are then given by

$$\begin{aligned}
S_{\text{inc}} &= \Re\left[-\frac{i\hbar}{m} u_{\text{inc}}^* \frac{du_{\text{inc}}}{dx}\right] = \frac{\hbar k |A|^2}{m} \\
S_{\text{trans}} &= \Re\left[-\frac{i\hbar}{m} u_{\text{trans}}^* \frac{du_{\text{trans}}}{dx}\right] = \frac{\hbar K |C|^2}{m} = \frac{4k^2}{(K+k)^2} \frac{\hbar K |A|^2}{m} \\
S_{\text{refl}} &= \Re\left[-\frac{i\hbar}{m} u_{\text{refl}}^* \frac{du_{\text{refl}}}{dx}\right] = -\frac{\hbar k |B|^2}{m} = -\frac{(K-k)^2}{(K+k)^2} \frac{\hbar k |A|^2}{m}
\end{aligned}$$

which according to the definitions above, Eqs. (3) and (4), gives the transmission and reflection coefficients

$$\begin{cases} T = \frac{4kK}{(K+k)^2} \\ R = \frac{(K-k)^2}{(K+k)^2}. \end{cases}$$

Note that $T + R = 1$ as it should be.

4. a) The Schrödinger equation in the region $-a \leq x \leq a$ looks like

$$\frac{d^2u}{dx^2} + \frac{2mE}{\hbar^2} u(x) = 0 \tag{5}$$

and the boundary conditions are $u(-a) = u(a) = 0$ since the potential is infinite for $x < -a$ and $x > a$. It is easily seen that the solutions given fulfill the boundary conditions and we then just have to check that they satisfy the Schrödinger equation with the given energy eigenvalues. For the even solutions we get

$$\frac{d^2 u_n}{dx^2} = \frac{d^2}{dx^2}\left(\sqrt{\frac{1}{a}} \cos \frac{n\pi x}{2a}\right) = -\frac{n^2\pi^2}{4a^2}\left(\sqrt{\frac{1}{a}} \cos \frac{n\pi x}{2a}\right) = -\frac{n^2\pi^2}{4a^2} u_n(x)$$

By putting this into Eq. (5) we see that it is satisfied if

$$-\frac{n^2\pi^2}{4a^2} = -\frac{2mE}{\hbar^2} \qquad \Rightarrow \qquad E = \frac{n^2\pi^2\hbar^2}{8ma^2},$$

i e for the given energy eigenvalue. We have thus proven that the even solutions given are solutions to the Schrödinger eqaution for the given potential. In the same way we can prove that the odd solutions given also are solutions to the Schrödinger equation.

Now let's prove that the even eigenfunctions are normalized,

$$\int_{-a}^{a} \cos^2 \frac{n\pi x}{2a} dx = \int_{-a}^{a} \left[\frac{1}{2} + \cos \frac{n\pi x}{a}\right] = 1$$

since the *cos* function is integrated over a whole number of periods. In the same way one shows that the odd eigenfunctions are normalized.

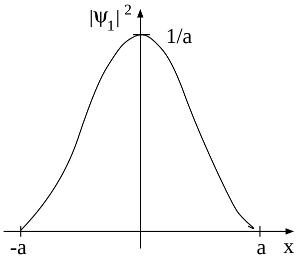
b) The probability distribution for a particle in the ground state is

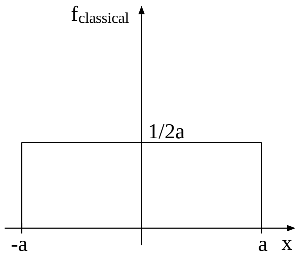
$$|\psi_1|^2 = \frac{1}{a} \cos^2 \frac{\pi x}{2a} \quad , \quad -a \leq x \leq a.$$

A *classical* particle can be anyware in the interval with equal probability so for a classical particle the probability distribution function is flat, i e

$$f_{\text{classical}} = \frac{1}{2a} \quad , \quad -a \leq x \leq a.$$

The probability distributions hence looks something like





c) The wave function at an arbitrary time t is given by

$$\begin{aligned}\Psi(x,t) &= \sqrt{\frac{1}{4}}\psi_1(x)e^{-iE_1t/\hbar} + \sqrt{\frac{3}{4}}\psi_2(x)e^{-iE_2t/\hbar} \\ &= \sqrt{\frac{1}{4a}}\cos\frac{\pi x}{2a}e^{-i\pi^2\hbar t/8ma^2} + \sqrt{\frac{3}{4a}}\sin\frac{\pi x}{a}e^{-i\pi^2\hbar t/2ma^2}\end{aligned}$$

since ψ_1 and ψ_2 are solutions to the time independent Schrödinger equation with energy eigenvalues E_1 and E_2.

d) The expectation value of x is given by

$$\begin{aligned}\langle x\rangle &= \int_{-a}^{a}\Psi^*(x,t)x\Psi(x,t)dx \\ &= \frac{1}{4}\int_{-a}^{a}\left[\psi_1(x)e^{iE_1t/\hbar} + \sqrt{3}\psi_2(x)e^{iE_2t/\hbar}\right]x\left[\psi_1(x)e^{-iE_1t/\hbar} + \sqrt{3}\psi_2(x)e^{-iE_2t/\hbar}\right]dx \\ &= \frac{1}{4}\int_{-a}^{a}\Big[\underbrace{\psi_1^2(x)x}_{\text{odd}} + 3\underbrace{\psi_2^2(x)x}_{\text{odd}} + \sqrt{3}\underbrace{\left(e^{-i(E_2-E_1)t/\hbar} + e^{i(E_2-E_1)t/\hbar}\right)}_{2\cos\frac{(E_2-E_1)t}{\hbar}}\psi_1(x)\psi_2(x)x\Big]dx \\ &= \frac{\sqrt{3}}{2}\cos\frac{(E_2-E_1)t}{\hbar}\int_{-a}^{a}\psi_1(x)\psi_2(x)xdx \\ &= \frac{\sqrt{3}}{2a}\cos\left[\frac{(E_2-E_1)t}{\hbar}\right]\int_0^a x\cos\frac{\pi x}{2a}\sin\frac{\pi x}{a}dx.\end{aligned}$$

Now let $q = \pi x/2a$. The expression above is then given by

$$\begin{aligned}\langle x\rangle &= \frac{\sqrt{3}}{2a}\cos\left[\frac{(E_2-E_1)t}{\hbar}\right]\left(\frac{2a}{\pi}\right)^2\int_{-\pi/2}^{\pi/2}q\cos q\sin 2q dq \\ &= \frac{\sqrt{3}2a}{\pi^2}\cos\left[\frac{(E_2-E_1)t}{\hbar}\right]\frac{8}{9} = \frac{16a\sqrt{3}}{9\pi^2}\cos\left[\frac{(E_2-E_1)t}{\hbar}\right]\end{aligned}$$

The expectation value of $\hat{p}$ is given by

$$\begin{aligned}\langle p\rangle &= \int_{-a}^{a}\Psi^*(x,t)\left(-i\hbar\frac{d}{dx}\right)\Psi(x,t)dx \\ &= -\frac{i\hbar}{4}\int_{-a}^{a}\left[\psi_1(x)e^{iE_1t/\hbar} + \sqrt{3}\psi_2(x)e^{iE_2t/\hbar}\right]\frac{d}{dx}\left[\psi_1(x)e^{-iE_1t/\hbar} + \sqrt{3}\psi_2(x)e^{-iE_2t/\hbar}\right]dx \\ &= -\frac{i\hbar}{4a}\int_{-a}^{a}\left[\cos\frac{\pi x}{2a}e^{iE_1t/\hbar} + \sqrt{3}\sin\frac{\pi x}{a}e^{iE_2t/\hbar}\right] \\ &\quad\times\left[-\frac{\pi}{2a}\sin\frac{\pi x}{2a}e^{-iE_1t/\hbar} + \sqrt{3}\frac{\pi}{a}\cos\frac{\pi x}{a}e^{-iE_2t/\hbar}\right]dx \\ &= -\frac{i\hbar\pi}{4a^2}\int_{-a}^{a}\Big[-\frac{1}{2}\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a} + 3\sin\frac{\pi x}{a}\cos\frac{\pi x}{a} \\ &\quad+\sqrt{3}\cos\frac{\pi x}{2a}\cos\frac{pix}{a}e^{-i(E_2-E_1)t/\hbar} - \frac{\sqrt{3}}{2}\sin\frac{pix}{2a}\sin\frac{pix}{a}e^{i(E_2-E_1)t/\hbar}\Big]dx \\ &= -\frac{i\hbar\pi}{4a^2}\int_{-a}^{a}\Big[-\frac{1}{4}\underbrace{\sin\frac{\pi x}{a}}_{\text{whole period}} + \frac{3}{2}\underbrace{\sin\frac{2\pi x}{a}}_{\text{whole period}}\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{3}\cos\frac{\pi x}{2a}\cos\frac{\pi x}{a}e^{-i(E_2-E_1)t/\hbar}-\frac{\sqrt{3}}{2}\sin\frac{\pi x}{2a}\sin\frac{\pi x}{a}e^{i(E_2-E_1)t/\hbar}\Big]dx \\
= \quad &-\frac{i\hbar\pi\sqrt{3}}{4a^2}\int_{-a}^{a}\left[\cos\frac{\pi x}{2a}\cos\frac{\pi x}{a}e^{-i(E_2-E_1)t/\hbar}-\frac{1}{2}\sin\frac{\pi x}{2a}\sin\frac{\pi x}{a}e^{i(E_2-E_1)t/\hbar}\right]dx
\end{aligned}
$$

Now let $q = \pi x/2a$. The expression above is then given by

$$
\begin{aligned}
\langle p\rangle = &-\frac{i\hbar\pi\sqrt{3}}{4a^2}\frac{2a}{\pi}\Big[e^{-i(E_2-E_1)t/\hbar}\int_{-\pi/2}^{\pi/2}\cos q\cos 2q dq \\
&-\frac{1}{2}e^{i(E_2-E_1)t/\hbar}\int_{-\pi/2}^{\pi/2}\sin q\sin 2q dq\Big] \\
= \quad &-\frac{i\hbar\sqrt{3}}{2a}\left[\frac{2}{3}e^{-i(E_2-E_1)t/\hbar}-\frac{2}{3}e^{i(E_2-E_1)t/\hbar}\right] \\
= \quad &-\frac{i\hbar\sqrt{3}}{3a}2i\sin\left[\frac{(E_2-E_1)t}{\hbar}\right] \\
= \quad &\frac{\hbar 2\sqrt{3}}{3a}\sin\left[\frac{(E_2-E_1)t}{\hbar}\right]
\end{aligned}
$$

The expectation value of E is given by

$$
\langle E\rangle = \frac{1}{4}E_1+\frac{3}{4}E_2 = \frac{1}{4}\frac{\pi^2\hbar^2}{8ma^2}+\frac{3}{4}\frac{\pi^2\hbar^2}{2ma^2} = \frac{13\pi^2\hbar^2}{32ma^2}
$$

since it cannot depend on time.

e) The correction to the ground state energy E_1 from the perturbation

$$
V'(x) = c(a-|x|)
$$

is in first order perturbation theory given by

$$
\begin{aligned}
\Delta E = &\int_{-a}^{a}\psi_1^*(x)V'(x)\psi_1(x)dx = 2\int_0^a\psi_1^*(x)V'(x)\psi_1(x)dx \\
= \quad &\frac{2c}{a}\int_0^a(a-x)cos^2\frac{\pi x}{2a}dx = \{\text{integration by parts}\} \\
= \quad &ca\left[\frac{1}{2}+\frac{2}{\pi^2}\right]
\end{aligned}
$$

5. a) Let $\hat{S}^2$ act on the state $\alpha_1\alpha_2$,

$$
\begin{aligned}
\hat{S}^2\alpha_1\alpha_2 = &\left(\hat{S}_1^2+\hat{S}_2^2+2\hat{S}_{1z}\hat{S}_{2z}+\hat{S}_{1+}\hat{S}_{2-}+\hat{S}_{1-}\hat{S}_{2+}\right)\alpha_1\alpha_2 \\
= \quad &\left(\frac{3}{4}\hbar^2+\frac{3}{4}\hbar^2+2\frac{1}{2}\hbar\frac{1}{2}\hbar+0+0\right)\alpha_1\alpha_2 \\
= \quad &2\hbar^2\alpha_1\alpha_2 = 1(1+1)\hbar^2\alpha_1\alpha_2.
\end{aligned}
$$

Hence $\alpha_1\alpha_2$ is an eigenstate of $\hat{S}^2$ with the total spin quantum number $s=1$. Now let $\hat{S}_z$ act on $\alpha_1\alpha_2$,

$$
\hat{S}_z\alpha_1\alpha_2 = \left(\hat{S}_{1z}+\hat{S}_{2z}\right)\alpha_1\alpha_2 = \left(\frac{1}{2}\hbar+\frac{1}{2}\hbar\right)\alpha_1\alpha_2 = \hbar\alpha_1\alpha_2.
$$

Hence $\alpha_1\alpha_2$ is also an eigenstate of $\hat{S}_z$ with the magnetic quantum number $m_s = 1$. The state $\alpha_1\alpha_2$ is therefore described by $|s,m_s\rangle = |1,1\rangle$.

b) Now apply the ladder operator $\hat{S}_-$ to the state $|1,1\rangle$,

$$
\hat{S}_-|1,1\rangle = \{\text{Physics Handbook}\} = \sqrt{(1+1)(1-1+1)}\hbar|1,0\rangle = \sqrt{2}\hbar|1,0\rangle. \tag{6}
$$

Now consider the same expression but let $\hat{S}_- = \hat{S}_{1-}\hat{S}_{2-}$ act directly on the states α_1 and α_2,

$$
\hat{S}_-|1,1\rangle = (\hat{S}_{1-}\hat{S}_{2-})\alpha_1\alpha_2 = \{\text{Physics Handbook}\} = \hbar(\beta_1\alpha_2+\alpha_1\beta_2). \tag{7}
$$

Putting Eqs. (6) and (7) equal then gives

$$|1,0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2)$$

To find the state $|0,0\rangle$ we note that it can only be made up of the states $\alpha_1\beta_2$ and $\beta_1\alpha_2$ because these are the only combinations having the magnetic quantum number $m_s = m_{s_1} + m_{s_2} = 0$. Hence let the state be

$$|0,0\rangle = a\alpha_1\beta_2 + b\beta_1\alpha_2$$

and use the fact that it should be orthogonal to all the other states and specifically to $|1,0\rangle$,

$$\begin{aligned}
0 = \langle 1,0|0,0\rangle &= \frac{1}{\sqrt{2}}\left(\langle\alpha_1\beta_2| + \langle\beta_1\alpha_2|\right)\left(a|\alpha_1\beta_2\rangle + b|\beta_1\alpha_2\rangle\right) \\
&= \frac{1}{\sqrt{2}}\Big[a\underbrace{\langle\alpha_1\beta_2|\alpha_1\beta_2\rangle}_{1} + b\underbrace{\langle\alpha_1\beta_2|\beta_1\alpha_2\rangle}_{0} + a\underbrace{\langle\beta_1\alpha_2|\alpha_1\beta_2\rangle}_{0} + b\underbrace{\langle\beta_1\alpha_2|\beta_1\alpha_2\rangle}_{1}\Big] \\
&= \frac{1}{\sqrt{2}}(a+b)
\end{aligned}$$

Further on we want the state $|0,0\rangle$ to be normalized, i e

$$\begin{aligned}
1 = \langle 0,0|0,0\rangle &= \left(a^*\langle\alpha_1\beta_2| + b^*\langle\beta_1\alpha_2|\right)\left(a|\alpha_1\beta_2\rangle + b|\beta_1\alpha_2\rangle\right) \\
&= \Big[a^*a\underbrace{\langle\alpha_1\beta_2|\alpha_1\beta_2\rangle}_{1} + a^*b\underbrace{\langle\alpha_1\beta_2|\beta_1\alpha_2\rangle}_{0} + ab^*\underbrace{\langle\beta_1\alpha_2|\alpha_1\beta_2\rangle}_{0} + bb^*\underbrace{\langle\beta_1\alpha_2|\beta_1\alpha_2\rangle}_{1}\Big] \\
&= |a|^2 + |b|^2.
\end{aligned}$$

We can thus choose a and b to be equal to $1/\sqrt{2}$ and $-1/\sqrt{2}$, i e

$$|0,0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$$

c) We can first note that the state χ is normalized and thus the probability of finding the z component of the spin being $+\frac{1}{2}\hbar$ is given by the absolute value squared of the coefficient in front of α, i e

$$P\left(+\frac{1}{2}\hbar\right) = \frac{2}{3}.$$

The expectation value $\langle S_z\rangle$ is then given by

$$\langle S_z\rangle = \frac{2}{3}\frac{1}{2}\hbar - \frac{1}{3}\frac{1}{2}\hbar = \frac{1}{6}\hbar$$

6. a) The potential can in centre of mass coordinates and relative coordinates be expressed as

$$V(X,x) = V_X(X) + V_x(x)$$

where

$$\begin{aligned}
V_X(X) &= \frac{1}{2}M\omega^2X^2 \\
V_x(x) &= \begin{cases} 0 & , \quad |x| < a \\ \infty & , \quad |x| \geq a \end{cases}
\end{aligned}$$

and the total mass $M = 2m$ is introduced.

b) The Hamilton operator,

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_X(X) + V_x(x)$$

can using the hint be written as

$$\hat{H} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + V_X(X) + V_x(x)$$

where the reduced mass

$$\mu = \frac{mm}{m+m} = \frac{m}{2}$$

has been introduced. Since the potential can be written as a sum of a function of X and a funcion of x we can do the Ansatz

$$\psi(X,x) = A(X)B(x)$$

which if we put it into the Schrödinger equation gives us two one dimensional Schrödinger equations

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} + V_X(X)\right] A(X) = E_X A(X) \tag{8}$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + V_x(x)\right] B(x) = E_x B(x) \tag{9}$$

where the total energy is given by $E = E_X + E_x$. The solutions of Eq. (8) are given by Physics Handbook

$$\begin{aligned} A_i(X) &= N_i H_i(\alpha X) e^{-\frac{1}{2}\alpha^2 X^2} \quad , \quad \alpha = \sqrt{\frac{M\omega}{\hbar}} \\ E_{Xi} &= \left(i + \frac{1}{2}\right)\hbar\omega \quad , \quad i = 0,1,2,3,\ldots \end{aligned}$$

where $H_i(\alpha X)$ are the Hermite polynomials. The solutions of Eq. (9) are given in problem 4)

$$\begin{aligned} \psi_j(x) &= \begin{cases} \sqrt{\frac{1}{a}}\cos\frac{j\pi x}{2a} & , \quad j = 1,3,5,7,\ldots \\ \sqrt{\frac{1}{a}}\sin\frac{j\pi x}{2a} & , \quad j = 2,4,6,8,\ldots \end{cases} \quad , \quad -a \le x \le a \\ E_{xj} &= \frac{j^2\pi^2\hbar^2}{8\mu a^2} \quad , \quad j = 1,2,3,4,\ldots \end{aligned}$$

Hence the total energy is given by

$$E_{ij} = \left(i + \frac{1}{2}\right)\hbar\omega + \frac{j^2\pi^2\hbar^2}{8\mu a^2} \quad , \quad \begin{cases} i = 0,1,2,3,\ldots \\ j = 1,2,3,4,\ldots. \end{cases} \tag{10}$$

c) i) For two non-identical particles all values of i and j given in Eq. (10) are possible.

ii) For two identical fermions with spin $\frac{1}{2}$ and the total spin being 1 we note that the spin wave function χ_{sm} is symmetric since the spins are parallell (the total spin is 1). We can also note that the function $A(X)$ is always symmetric with respect to exchange of particle 1 and 2 since $X \to X$ under such an exchange. However, $x \to -x$ under exchange of particle 1 and 2 and hence $B_j(x)$ is symmetric when j is odd and antisymmetric when j is even. The only way to get the total wave function antisymmetric in this case is thus to have $B_j(x)$ antisymmetric, i e j even. Hence the possible energy eigenvalues are

$$E_{ij} = \left(i + \frac{1}{2}\right)\hbar\omega + \frac{j^2\pi^2\hbar^2}{8\mu a^2} \quad , \quad \begin{cases} i = 0,1,2,3,\ldots \\ j = 2,4,6,\ldots. \end{cases}$$

iii) For two bosons with spin 0 the spin wave function is of course symmetric and following the discussion above we see that the only way to get the total wave function symmetric is to have $B_j(x)$ symmetric, i e j odd. Hence the possible energy eigenvalues are

$$E_{ij} = \left(i + \frac{1}{2}\right)\hbar\omega + \frac{j^2\pi^2\hbar^2}{8\mu a^2} \quad , \quad \begin{cases} i = 0,1,2,3,\ldots \\ j = 1,3,5,\ldots. \end{cases}$$