F3EPT
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# Examination in Quantum Mechanics

**11 October 1996**

To be used: Mathematics Handbook, Physics Handbook and pocket calculator.

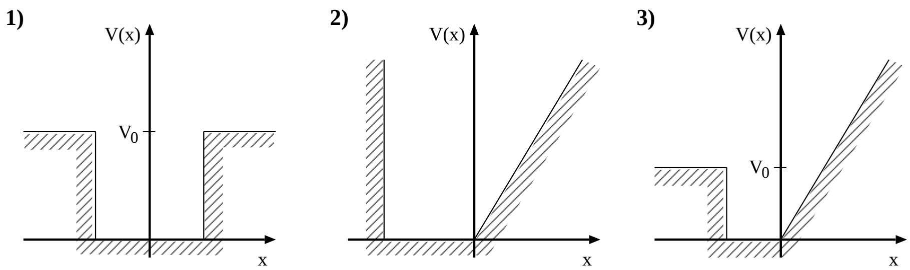
**6 problems in 6 hours**.
Each problem gives a maximum of 5 points.
Read the problems through before you start solving them.
Please start each solution on a new paper.

**1.** a) Show that if two operators $\hat{A}$ and $\hat{B}$ have a common set of eigenfunctions then $[\hat{A}, \hat{B}] = 0$.

b) Consider only the case that $\hat{A}$ has non-degenerate eigenvalues and show that if $\hat{A}$ and $\hat{B}$ commute they possess a common set of eigenfunctions.

c) Show that the eigenvalues of a Hermitian operator are real.

**2.** a) Consider the following potentials,




i) Which potentials are invariant under parity ($x \to -x$)?

ii) What can you say about the parity of the energy eigenfunctions for the different potentials?

iii) For each potential, state if it will have discrete or continous energy spectrum (or both).

b) Consider the following three-dimensional potentials

1) $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$
2) $V(\mathbf{r}) = ax^2 + by^4 + cz^2$
3) $V(\mathbf{r}) = ax^2 + by^4$
4) $V(\mathbf{r}) = \frac{a}{r^2}$

where $a$, $b$, and $c$ are constants and $r = \sqrt{x^2 + y^2 + z^2}$.

i) Which potentials are separable in spherical coordinates?

ii) Which potentials are separable in cartesian coordinates?

iii) Which potentials are invariant under parity ($\mathbf{r} \to -\mathbf{r}$)?

c) Consider the following time independent wave functions describing a system of two particles in one dimension

$$\begin{array}{lll} 1) & \psi = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\chi_{sm} & , \quad 0 < x_1, x_2 < a \\ 2) & \psi = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\chi_{sm} & , \quad 0 < x_1, x_2 < a \\ 3) & \psi = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right]\chi_{sm} & , \quad 0 < x_1, x_2 < a \end{array}$$

where $x_1$ is the position of particle 1, $x_2$ is the position of particle 2 and $\chi_{sm}$ is the spin wave function for the two particles.

i) Which wave functions are suitable for two non-identical particles?

ii) Which wave functions are suitable for two identical fermions if $\chi_{sm}$ is symmetric with respect to exchange of particle 1 and 2?

iii) Which wave functions are suitable for two identical bosons if $\chi_{sm}$ is symmetric with respect to exchange of particle 1 and 2?

d) Consider a hydrogen atom (neglecting spin) with the stationary states $\psi_{nlm}$ and energy eigenvalues $E_n = -\frac{\hbar^2}{2\mu a_0^2 n^2}$ where $\mu$ is the reduced mass of the nucleus and the electron and $a_0$ is the Bohr radius.

i) What is the degeneracy of the state with $n = 3$?

ii) A measurement of the $z$ component of the orbital angular momentum of an electron results in $2\hbar$. Which is the lowest energy state that the electron could be in?

iii) How much energy is required to excite this electron into the closest higher energy state?

**3.** A flow of particles (each with mass $m$) is incident from $x = -\infty$ on the potential step

$$V(x) = \begin{cases} 0 & , \quad x < 0 \\ V_0 & , \quad x > 0 \end{cases}$$

Define the transmission and reflection coefficients $T$ and $R$ and derive expressions for them when $E > V_0$.

**4.** Consider a particle of mass $m$ in the potential

$$V(x) = \begin{cases} \infty & , \quad x < -a \\ 0 & , \quad -a < x < a \\ \infty & , \quad x > a. \end{cases}$$

a) Write down the Schrödinger equation and the boundary conditions for the solutions and show that the normalized time independent eigenfunctions in the region $-a \leq x \leq a$ are given by

$$\psi_n(x) = \begin{cases} \sqrt{\frac{1}{a}}\cos\frac{n\pi x}{2a} & , \quad n = 1, 3, 5, 7, \ldots \\ \sqrt{\frac{1}{a}}\sin\frac{n\pi x}{2a} & , \quad n = 2, 4, 6, 8, \ldots \end{cases} \quad , \quad -a \leq x \leq a$$

with energy eigenvalues $E_n = \frac{n^2\pi^2\hbar^2}{8ma^2}$.

b) Sketch the probability distribution for a particle in the ground state $\psi_1(x)$. Also sketch the probability distribution for a *classical* particle in this potential.

c) At $t = 0$ the wave function for a particle in this potential is given by

$$\Psi(x,0) = \sqrt{\frac{1}{4}}\psi_1(x) + \sqrt{\frac{3}{4}}\psi_2(x).$$

Write down the wave function $\Psi(x,t)$ at an arbitrary time $t$.

d) For the wave function $\Psi(x,t)$ found in c), determine the expectation values $\langle x\rangle$, $\langle p\rangle$ and $\langle E\rangle$ as functions of time.
Hint:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q\cos q \sin 2q dq = \frac{8}{9}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos q \cos 2q dq = \frac{2}{3}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin q \sin 2q dq = \frac{4}{3}$$

e) Consider a small perturbation

$$V'(x) = c(a - |x|)$$

where $c$ is a small constant and determine the correction to the ground state energy using first order perturbation theory.

**5.** Consider two particles with spin $s_1 = s_2 = \frac{1}{2}$. Let $\alpha_1$ and $\beta_1$ be the eigenfuncions of $\hat{S}_1^2$ and $S_{1z}$ corresponding to spin up ($m_{s_1} = +\frac{1}{2}$) and spin down ($m_{s_1} = -\frac{1}{2}$) for the first particle. Similary let $\alpha_2$ and $\beta_2$ be the eigenfuncions of $\hat{S}_2^2$ and $S_{2z}$ corresponding to spin up ($m_{s_2} = +\frac{1}{2}$) and spin down ($m_{s_2} = -\frac{1}{2}$) for the second particle.

a) Show that $\alpha_1\alpha_2$ is the common eigenfunction of $\hat{S}^2$ and $\hat{S}_z$ where $\hat{S} = \hat{S}_1 + \hat{S}_2$. Also determine the corresponding quantum numbers $s$ and $m_s$ for this state where $s(s+1)\hbar^2$ and $m_s\hbar$ are the eigenvalues of $\hat{S}^2$ and $\hat{S}_z$.
Hint: $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$ where $\hat{S}_{1\pm}$ and $\hat{S}_{2\pm}$ are ladder operators.

b) Find the common eigenfunction of $\hat{S}^2$ and $\hat{S}_z$ corresponding to the quantum numbers $|s, m_s\rangle = |1, 0\rangle$ and $|s, m_s\rangle = |0, 0\rangle$.

c) Consider the spin state of a spin-$\frac{1}{2}$ particle

$$\chi = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$$

in the representation where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

What is the probability of finding the $z$ component of the spin being $+\frac{1}{2}\hbar$ and what is the expectation value $\langle S_z\rangle$?

**6.** Two particles (mass $m$) are restrained to move in one dimension in the potential

$$V(x_1, x_2) = \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2$$

where index 1 denotes particle 1 and index 2 denotes particle 2. In addition to the potential above there is an interaction between the particles described by the potential

$$V_{\text{int}}(x_1, x_2) = \begin{cases} -\frac{1}{4}m\omega^2(x_1 - x_2)^2 & , \quad |x_1 - x_2| < a \\ \infty & , \quad |x_1 - x_2| \geq a. \end{cases}$$

a) Express the potential in centre of mass coordinates and relative coordinates,

$$\begin{aligned} X &= \frac{mx_1 + mx_2}{2m} = \frac{x_1 + x_2}{2} \\ x &= x_1 - x_2 \end{aligned}$$

b) The total wave function describing the system can be written

$$\Psi = \psi(X, x)\chi_{sm}$$

where $\chi_{sm}$ is an eigenfuncion of $\hat{S}^2$ and $\hat{S}_z$ ($\hat{S} = \hat{S}_1 + \hat{S}_2$). Using that

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{1}{2}\frac{\partial^2}{\partial X^2} + 2\frac{\partial^2}{\partial x^2}$$

solve the time independent Schrödinger equation for this system to obtain general expressions for the energy eigenvalues and eigenfunctions. Physics Handbook and the information given in problem 4) may be used.

c) What are the possible energy eigenvalues if

i) the two particles are non-identical?

ii) the two particles are identical fermions with spin $\frac{1}{2}$ and the total spin being 1?

iii) the two particles are identical bosons with spin 0?

**GOOD LUCK!**

Solutions will be available on `http://www.teorfys.uu.se/~edsjo/kmek/index.html` as well as posted at Institutionen för teoretisk fysik directly after the examination.