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Examination in Quantum Mechanics

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To be used: Mathematics Handbook, Physics Handbook and pocket calculator.

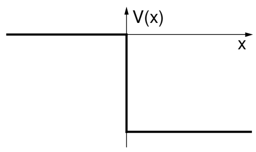
5 problems in 6 hours.
Each problem gives a maximum of 5 points.
Read the problems through before you start solving them.
Please start each solution on a new paper.

1. a) Derive the Ehrenfest theorem for a particle with mass m in a real potential $V(\mathbf{r})$.

b) Prove that two eigenstates of a hermitian operator with different eigenvalues are orthogonal.

2. a) Particles each with mass m and energy $E = \hbar^2/2ma^2$ are steadily flowing from the left ($x = -\infty$) towards a one-dimensional potential step

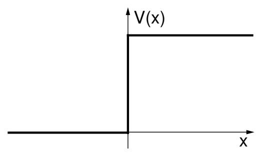
$$V(x) = \begin{cases} 0 & , \quad x < 0 \\ -\frac{3\hbar^2}{2ma^2} & , \quad x > 0 \end{cases}$$



where a is a constant of dimension length. Define the transmission coefficient T and the reflection coefficient R. Determine these quantities from their respective definitions.

b) Consider a steady flow of particles with mass m and energy $E = \hbar^2/2ma^2$ towards the one-dimensional potential step

$$V(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{3\hbar^2}{2ma^2} & , \quad x > 0. \end{cases}$$



Determine the wave function $u(x)$ (apart from normalization) in the regions $x < 0$ and $x > 0$ assuming that it is real and positive for $x > 0$. (You may put $u(0) = 1$.)

3. A particle (mass m, energy E and spin neglected) in a central potential $V(r)$ is in a stationary state given by the (time independent) wave function in spherical coordinates

$$\psi(r, \theta, \varphi) = \frac{1}{8\sqrt{\pi}a^{5/2}} r e^{-r/2a} \sin\theta e^{i\varphi}.$$

a) Show that the wave function is normalized. Show that the orbital angular momentum quantum number is 1. What is the magnetic quantum number?

b) Show that $E = -\hbar^2/8ma^2$ and determine $V(r)$ provided $V(r) \to 0$ as $r \to \infty$.

c) Determine the probability current density at any point (r, θ, φ) and determine the total probability current through the half plane $\varphi = 0$. Show that this total probability current is of the same order of magnitude as $|E|/h$, h being the unbarred Planck's constant.
Hints:

$$\hat{L}_x + i\hat{L}_y = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

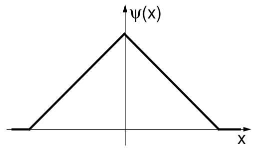
$$\int_0^{\pi} (\sin\theta)^{2l+1} d\theta = 2^{2l+1}\frac{(l!)^2}{(2l+1)!}$$

$$\int_0^{\infty} r^k e^{-r/a} dr = k!\, a^{k+1}$$

The direction of the probability current density is orthogonal to the half plane $\varphi =$ constant, i e it is directed in the $\mathbf{e}_\varphi$ direction.

4. The motion of a particle is at a certain instant represented by the one-dimensional normalized wave packet (a being a constant of the same dimension as x)

$$\psi(x) = \begin{cases} \sqrt{\frac{3}{2}}\frac{a-|x|}{a^{3/2}} & , \quad -a < x < a \\ 0 & , \quad |x| > a. \end{cases}$$



a) Calculate the uncertainty (standard deviation) in position Δx.

b) Calculate the uncertainty (standard deviation) in momentum Δp_x.

c) Check the Heisenberg uncertainty principle.

Hints: The wave function in momentum space can be used in b),

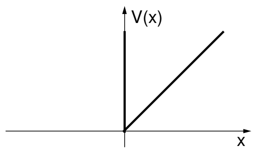
$$\phi(p_x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ixp_x/\hbar}\psi(x)dx.$$

Alternatively one may use the hermiticity of the momentum operator to avoid letting the momentum operator act twice upon the same wave function.

$$\int_{-\infty}^{\infty}\frac{[1-\cos(pc)]^2}{p^2}dp = \pi c \quad , \quad c \text{ being a constant}$$

5. **a)** A particle with mass m moves in the potential well

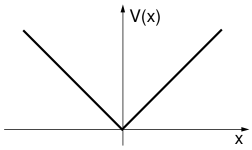
$$V(x) = \begin{cases} \infty & , \quad x < 0 \\ mgx & , \quad x > 0 \end{cases}$$



where g is a constant (dimension of acceleration). Determine the energy eigenvalues ($E > 0$) using the hint below.

b) Consider a particle with mass m moving in the potential

$$V(x) = mg|x| \quad , \quad -\infty < x < \infty.$$



Show that all the energy eigenvalues determined in a) are energy eigenvalues for the last mentioned potential as well. Are there further energy eigenvalues? Notice that problem b) can be solved even if you have not done problem a).

Hints: The differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi(x) = E\psi(x) \tag{1}$$

has the solution (apart from normalization) for all values of x, i e for $-\infty < x < \infty$

$$\psi(x) = z^{1/3}\left(J_{-1/3}(z) + J_{1/3}(z)\right) \quad \text{with } z = \beta(E - mgx)^{3/2} \text{ and } \beta = \frac{2\sqrt{2}}{3g\hbar\sqrt{m}}$$

where $J_\mu(z)$ denotes a Bessel function of order μ.

For $x > E/mg$, z becomes imaginary and it is more practical to express ψ in an alternative (equivalent) way as essentially a modified Bessel function,

$$\psi(x) = \frac{\sqrt{3}}{\pi}|z|^{1/3}K_{1/3}(|z|) \quad , \quad |z| = \beta(mgx - E)^{3/2}$$

For $x \gg E/mg$, ψ behaves as

$$\psi(x) \simeq A\exp\left(-\beta(mgx - E)^{3/2}\right),$$

where A is a decreasing function of x. Any other linearly independent solution of Eq. (1) is exponentially increasing (essentially as $\exp\left(+\beta(mgx - E)^{3/2}\right)$) for $x \gg E/mg$. For any value of $E > 0$, if x is less than and not too close to E/mg, ψ may be approximated by

$$\psi(x) \simeq C\cos\left(\beta(E - mgx)^{3/2} - \frac{\pi}{4}\right) \quad , x \text{ less than and not too close to } E/mg,$$

where C is a slowly varying function of x but can be treated as a constant here. For determining the energy eigenvalues the last mentioned approximation of ψ can be used as if it were "exact" in the classically allowed region apart from the vicinity of $x = E/mg$.

GOOD LUCK!

Solutions will be available on `http://www.teorfys.uu.se/~edsjo/kmek/index.html` as well as posted at Institutionen för teoretisk fysik directly after the examination.