Solutions to exam in Quantum Mechanics

12 October 1995

Solutions made by Joakim Edsjö

1. **a)** See Bransden & Joachain, pages 84–86.

b) See Bransden & Joachain, pages 114-115.

2. **Part 1. a)** The potential (a) is invariant under parity.

b) The potential (a) and (b) have only discrete energy spectrum, whereas potential (c) will have discrete energy spectrum for energies below V_1 and has continuous energy spectrum for energies above V_1.

c) Since the potential (a) is symmetric, the solutions will have definite parity and we need only consider $x > 0$ when solving the Schrödinger equation. For $x > 0$ the potential (a) is identical to (b) and thus the solutions are the same with the only difference being the continuity condition at $x = 0$ which for potential (b) restricts the solutions to only those where the wave function vanishes in $x = 0$, i e odd solutions. Hence all solutions of (b) are also solutions to (a).

Part 2. a) For potential (a) the Schrödinger equation is separable in Cartesian coordinates.

b) For potential (b) the Schrödinger equation is separable in spherical coordinates.

c) The potentials (a) and (b) are invariant under parity.

d) The orbital angular momentum $\vec{L}$ commutes with the Hamiltonian H for the potential (b) since $\vec{L}$ commutes with p^2 and the potential (b).

3. **a)** $\psi_1(x)$ and $\psi_3(x)$ are given by (see e g P.H. F-6.3 with a suitable transformation of variables)

$$\begin{aligned}\psi_1(x) &= \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}\\ \psi_3(x) &= \frac{1}{\sqrt{a}}\cos\frac{3\pi x}{2a}\end{aligned}$$

b) Since the parity of both $\psi_1(x)$ and $\psi_3(x)$ is even, the parity of $\psi(x)$ is even.

c) The energy of state n is given by

$$E_n = \frac{\pi^2\hbar^2 n^2}{8ma^2}$$

Since $\psi_1(x)$ and $\psi_3(x)$ are normalized, if we measure the energy we can get

$$\begin{aligned}E_1 &= \frac{\pi^2\hbar^2}{8ma^2} \quad \text{with probability} \quad \frac{2}{3}\\ E_3 &= \frac{9\pi^2\hbar^2}{8ma^2} \quad \text{with probability} \quad \frac{1}{3}\end{aligned}$$

d) Since $\psi(x)$ is even, the integrand in $< x >$ is odd and we integrate over a symmetric interval the integral in $< x >$ gives zero:

$$< x > = \int_{-a}^{a} \underbrace{\psi(x)^* x \psi(x)}_{\text{odd}} \, dx = 0$$

The expectation value of p is given by

$$< p > = \int_{-a}^{a} \underbrace{\psi(x)^*}_{\text{even}} \underbrace{\left(-i\hbar \frac{d}{dx}\right) \psi(x)}_{\text{odd}} \, dx = 0$$

The expectation value of the energy is equal to

$$< E > = \frac{2}{3} E_1 + \frac{1}{3} E_2 = \frac{11\pi^2\hbar^2}{24ma^2}$$

e) The expectation value $< x^2 >$ is given by

$$\begin{aligned} < x^2 > &= \int_{-a}^{a} \psi(x)^* x^2 \psi(x) dx \\ &= \frac{1}{a} \int_{-a}^{a} \left(\sqrt{\frac{2}{3}} \cos \frac{\pi x}{2a} + \sqrt{\frac{1}{3}} \cos \frac{3\pi x}{2a} \right) x^2 \left(\sqrt{\frac{2}{3}} \cos \frac{\pi x}{2a} + \sqrt{\frac{1}{3}} \cos \frac{3\pi x}{2a} \right) dx \end{aligned}$$

Now let $y = \frac{\pi x}{2a}$. Since the integrand is even we can integrate just over $x > 0$ and multiply by 2. This gives

$$\begin{aligned} < x^2 > &= \frac{2}{a} \left(\frac{2a}{\pi}\right)^3 \int_0^{\pi/2} \left(\sqrt{\frac{2}{3}} \cos y + \sqrt{\frac{1}{3}} \cos 3y \right) y^2 \left(\sqrt{\frac{2}{3}} \cos y + \sqrt{\frac{1}{3}} \cos 3y \right) dy \\ &= \frac{16a^2}{\pi^3} \int_0^{\pi/2} \left(\frac{2}{3} y^2 \cos^2 y + \frac{1}{3} y^2 \cos^2 3y + \frac{2\sqrt{2}}{3} y^2 \cos y \cos 3y \right) dy \end{aligned}$$

The hint then gives

$$\begin{aligned} < x^2 > &= \frac{16a^2}{\pi^3} \left(\frac{2}{3} \frac{\pi(\pi^2 - 6)}{48} + \frac{1}{3} \frac{\pi(9\pi^2 - 6)}{48 \cdot 9} - \frac{2\sqrt{2}}{3} \frac{3\pi}{32} \right) \\ &= \frac{a^2}{3} - \frac{a^2}{\pi^2} \frac{38 + 27\sqrt{2}}{27} \end{aligned}$$

The expectation value of p^2 is easiest calculated by noting that $p^2 = 2mE$. Hence

$$< p^2 > = < 2mE > = 2m < E > = 2m \frac{11\pi^2\hbar^2}{24ma^2} = \frac{11\pi^2\hbar^2}{12a^2}$$

f) Δ_x and Δ_p are given by

$$\begin{aligned} \Delta_x &= \sqrt{< (x - < x >)^2 >} = \sqrt{< x^2 > - < x >^2} = a \sqrt{\frac{1}{3} - \frac{1}{\pi^2} \frac{38 + 27\sqrt{2}}{27}} \\ \Delta_p &= \sqrt{< (p - < p >)^2 >} = \sqrt{< p^2 > - < p >^2} = \frac{\pi\hbar}{a} \sqrt{\frac{11}{12}} \end{aligned}$$

and thus we have

$$\Delta_x \Delta_p = a\sqrt{\frac{1}{3} - \frac{1}{\pi^2}\frac{38+27\sqrt{2}}{27}}\frac{\pi\hbar}{a}\sqrt{\frac{11}{12}} = \pi\hbar\sqrt{\frac{11}{12}}\sqrt{\frac{1}{3} - \frac{1}{\pi^2}\frac{38+27\sqrt{2}}{27}}$$

which is greater than $\hbar/2$ as it should.

4. **a)** The energy of a state ψ_{nlm} $(0 \leq l \leq n-1)$ is given by (P.H. F-6.4)

$$E_n = -\frac{\mu e^4 Z^2}{8\varepsilon_0^2 h^2 n^2}$$

and hence only depends on n. If we neglect spin we have for each n

$$\sum_{l=0}^{n-1}(2l+1) = \frac{n(1+2(n-1)+1)}{2} = n^2$$

different states with the same n and hence with the same energy. In our case we then have $2^2 = 4$ different states with the same energy.

b) To excite the particle to a $l = 3$ state requires that we excite the particle to a state with $n \geq 4$. The minimum energy required for the excitation is given by

$$E_{min} = -\frac{\mu e^4 Z^2}{8\varepsilon_0^2 h^2 4^2} + \frac{\mu e^4 Z^2}{8\varepsilon_0^2 h^2 2^2} = \frac{3\mu e^4 Z^2}{128\varepsilon_0^2 h^2}$$

c) We have the state

$$\psi = A\psi_{100} + B\psi_{322}$$

and we know that

$$\begin{aligned} L_z\psi_{100} &= 0\hbar\psi_{100} = 0 \\ L_z\psi_{322} &= 2\hbar\psi_{322} \end{aligned}$$

Hence the expectation value of L_z is given by

$$< L_z > = \; 2\hbar|B|^2$$

$< L_z > < \; \hbar \Rightarrow$

$$\begin{aligned} &2\hbar|B|^2 < \hbar \\ &\Rightarrow \; |B|^2 < \frac{1}{2} \end{aligned}$$

The expectation value of the energy is given by

$$< E > = \; |A|^2 E_1 + |B|^2 E_3$$

Now use that

$$|A|^2 + |B|^2 = 1$$
$$E_1 = 4E_2$$
$$E_3 = \frac{4}{9}E_2$$

which yields

$$\begin{aligned} <E> &= (1-|B|^2)E_1 + |B|^2E_3 = (1-|B|^2)4E_2 + |B|^2\frac{4}{9}E_2 \\ &= \left(4 - \frac{32}{9}|B|^2\right)E_2 \end{aligned}$$

Remembering that $E_2 < 0$ we see that $|B|^2 < \frac{1}{2} \Rightarrow$

$$\begin{aligned} <E> &< \left(4 - \frac{32}{9}\frac{1}{2}\right)E_2 \\ \Rightarrow <E> &< \frac{20}{9}E_2 \; < \; E_2 \end{aligned}$$

d) The magnetic field shifts the energy with

$$\Delta E = -\frac{eBL_z}{2\mu}$$

The state with $l = 1$ with lowest energy will be ψ_{211} (since $L_z\psi_{211} = \hbar\psi_{211}$ is maximal then) with energy

$$E_{211} = -\frac{\mu e^4 Z^2}{32\varepsilon_0^2 h^2} - \frac{eB\hbar}{2\mu}$$

The ψ_{100} state has the energy

$$E_{100} = -\frac{\mu e^4 Z^2}{8\varepsilon_0^2 h^2}$$

Requiring that $E_{211} < E_{100}$ yields

$$B > \frac{3\mu^2 e^3 Z^2}{64\pi^2\varepsilon_0^2\hbar^2}$$

However, since the energy goes as $\frac{1}{n^2}$ the lowest energy states with higher l will be even lower in energy, e g

$$E_{322} = -\frac{\mu e^4 Z^2}{72\varepsilon_0^2 h^2} - \frac{2eB\hbar}{2\mu}$$

which will always be lower than E_{211} when B is greater than the value calculated above. Hence it will never be possible to obtain a ground state with $l = 1$ by applying a strong enough magnetic field.

5. a) Consider

$$\int_{-\infty}^{\infty} |\varphi(x,t)|^2 dx = \int_{-\infty}^{\infty} |e^{iax+ibt}\psi(x-vt,t)|^2 dx$$
$$= \int_{-\infty}^{\infty} e^{-iax-ibt}\psi^*(x-vt,t)e^{iax+ibt}\psi(x-vt,t)dx$$
$$= \{x' = x - vt\} = \int_{-\infty}^{\infty} \psi^*(x',t)\psi(x',t)dx'$$
$$= \int_{-\infty}^{\infty} |\psi(x',t)|^2 dx' = 1$$

i e $\varphi(x,t)$ is normalized.

b) Ehrenfest's theorem states that (P.H. F-6.1)

$$m\frac{d}{dt} < x > = < p >$$
$$\frac{d}{dt} < p > = -\langle \frac{dV(x)}{dx} \rangle$$

Since $V(x) \equiv 0$ we get from the second equation above

$$\frac{d}{dt} < p > = 0 \qquad \Rightarrow \qquad < p > = p_0$$

where p_0 is a constant. From the first of Ehrenfest's equations we then get

$$\frac{d}{dt} < x > = \frac{1}{m} < p > = \frac{p_0}{m} \qquad \Rightarrow \qquad < x > = x_0 + \frac{p_0 t}{m}$$

where x_0 is a constant.

c) Insert $\varphi(x,t)$ into the Schrödinger equation.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x,t) = i\hbar\frac{\partial}{\partial t}\varphi(x,t)$$
$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{iax+ibt}\psi(x-vt,t)\right) = i\hbar\frac{\partial}{\partial t}\left(e^{iax+ibt}\psi(x-vt,t)\right)$$

Now let $x' = x - vt$ and consider

$$\frac{\partial^2}{\partial x^2}\left(e^{iax+ibt}\psi(x-vt,t)\right)$$
$$= \frac{\partial}{\partial x}\left(iae^{iax+ibt}\psi(x-vt,t) + e^{iax+ibt}\frac{\partial\psi(x',t)}{\partial x'}\underbrace{\frac{\partial x'}{\partial x}}_{1}\right)$$
$$= \left(-a^2 e^{iax+ibt}\psi(x-vt,t) + 2iae^{iax+ibt}\frac{\partial\psi(x',t)}{\partial x'} + e^{iax+ibt}\frac{\partial^2\psi(x',t)}{\partial x'^2}\right)$$
$$= \left(-a^2\psi(x-vt,t) + 2ia\frac{\partial\psi(x',t)}{\partial x'} + \frac{\partial^2\psi(x',t)}{\partial x'^2}\right)e^{iax+ibt}$$

For the time derivative we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(e^{iax+ibt}\psi(x-vt,t)\right)\\
&= ibe^{iax+ibt}\psi(x-vt,t)+e^{iax+ibt}\frac{\partial\psi(x',t)}{\partial x'}\underbrace{\frac{\partial x'}{\partial t}}_{-v}+e^{iax+ibt}\frac{\partial\psi(x',t)}{\partial t}\\
&= \left(ib\psi(x-vt,t)-v\frac{\partial\psi(x',t)}{\partial x'}+\frac{\partial\psi(x',t)}{\partial t}\right)e^{iax+ibt}
\end{aligned}$$

Put this into the Schrödinger equation above and we get

$$\begin{aligned}
&\left[\frac{\hbar^2}{2m}a^2\psi(x',t)-\frac{ia\hbar^2}{m}\frac{\partial\psi(x',t)}{\partial t}\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x',t)}{\partial x'^2}-i\hbar\frac{\partial\psi(x',t)}{\partial t}}_{0\ since\ \psi\ satisfies\ SE}\right.\\
&\left.+b\hbar\psi(x',t)+i\hbar v\frac{\partial\psi(x',t)}{\partial x'}\right]e^{iax+ibt}=0\\
\Rightarrow\ &\left[\left(\frac{\hbar^2a^2}{2m}+b\hbar\right)\psi(x',t)+i\left(v\hbar-\frac{a\hbar^2}{m}\right)\frac{\partial\psi(x',t)}{\partial x'}\right]e^{iax+ibt}=0
\end{aligned}$$

Since $\psi(x',t)$ and $\frac{\partial\psi(x',t)}{\partial x'}$ can be considered being independent functions, the coefficients in front of each one of them must vanish, i e

$$\left\{\begin{array}{l}\frac{\hbar^2a^2}{2m}+b\hbar=0\\ v\hbar-\frac{a\hbar^2}{m}=0\end{array}\right.\qquad\Rightarrow\qquad\left\{\begin{array}{l}a=\frac{mv}{\hbar}\\ b=-\frac{mv^2}{2\hbar}\end{array}\right.$$

Hence we get the function $\varphi(x,t)$ to be

$$\varphi(x,t)=e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi(x-vt,t)$$

d) Now consider the expectation value

$$\begin{aligned}
<\varphi|p|\varphi> &= \int_{-\infty}^{\infty}\varphi^*(x,t)p\varphi(x,t)dx\\
&= \int_{-\infty}^{\infty}e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)pe^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi(x-vt,t)dx\\
&= \int_{-\infty}^{\infty}e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)\left(-i\hbar\frac{\partial}{\partial x}e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\right)\psi(x-vt,t)dx\\
&\quad+\int_{-\infty}^{\infty}e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}p\psi(x-vt,t)dx\\
&= mv\int_{-\infty}^{\infty}\psi^*(x-vt,t)\psi(x-vt,t)dx+\int_{-\infty}^{\infty}\psi^*(x-vt,t)p\psi(x-vt,t)dx\\
&= \{x'=x-vt\}=mv\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)\psi(x',t)dx'}_{1}+\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)p\psi(x',t)dx'}_{<\psi|p|\psi>\ =\ p_0}\\
&= mv+p_0
\end{aligned}$$

The expectation value $<\varphi|x|\varphi>$ is given by

$$
\begin{aligned}
<\varphi|x|\varphi> &= \int_{-\infty}^{\infty} \varphi^*(x,t)x\varphi(x,t)dx \\
&= \int_{-\infty}^{\infty} e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)xe^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi(x-vt,t)dx \\
&= \{x'=x-vt\} = \int_{-\infty}^{\infty}\psi^*(x',t)(x'+vt)\psi(x',t)dx' \\
&= \underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)x'\psi(x',t)dx'}_{<\psi|x|\psi>=x_0+\frac{p_0t}{m}} + vt\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)\psi(x',t)dx'}_{1} \\
&= x_0+\frac{p_0t}{m}+vt
\end{aligned}
$$

We thus see that the transformation given in the problem corresponds to a Galilean boost with velocity v along the x-axis.

e) We now know that

$$
<\psi|\frac{p^2}{2m}|\psi> = E_0
$$

Consider

$$
\begin{aligned}
<\varphi|\frac{p^2}{2m}|\varphi> &= \int_{-\infty}^{\infty}\varphi^*(x,t)\frac{p^2}{2m}\varphi(x,t)dx \\
&= \int_{-\infty}^{\infty} e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi(x-vt,t)dx \\
&= \int_{-\infty}^{\infty} e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\right)\psi(x-vt,t)dx \\
&\quad+\int_{-\infty}^{\infty} e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)\left(-\frac{i\hbar}{m}\frac{\partial}{\partial x}e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\right)\left(-i\hbar\frac{\partial}{\partial x}\psi(x-vt,t)\right)dx \\
&\quad+\int_{-\infty}^{\infty} e^{-i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\psi^*(x-vt,t)e^{i\left(mvx-\frac{mv^2}{2}t\right)/\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x-vt,t)\right)dx \\
&= \frac{mv^2}{2}\int_{-\infty}^{\infty}\psi^*(x-vt,t)\psi(x-vt,t)dx + v\int_{-\infty}^{\infty}\psi^*(x-vt,t)p\psi(x-vt,t)dx \\
&\quad+\int_{-\infty}^{\infty}\psi^*(x-vt,t)\frac{p^2}{2m}\psi(x-vt,t)dx \\
&= \{x'=x-vt\} = \frac{mv^2}{2}\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)\psi(x',t)dx'}_{1} \\
&\quad+v\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)p\psi(x',t)dx'}_{<\psi|p|\psi>\,=\,p_0}+\underbrace{\int_{-\infty}^{\infty}\psi^*(x',t)\frac{p^2}{2m}\psi(x',t)dx'}_{<\psi|\frac{p^2}{2m}|\psi>\,=\,E_0} \\
&= \frac{mv^2}{2}+vp_0+E_0
\end{aligned}
$$