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Examination in Quantum Mechanics

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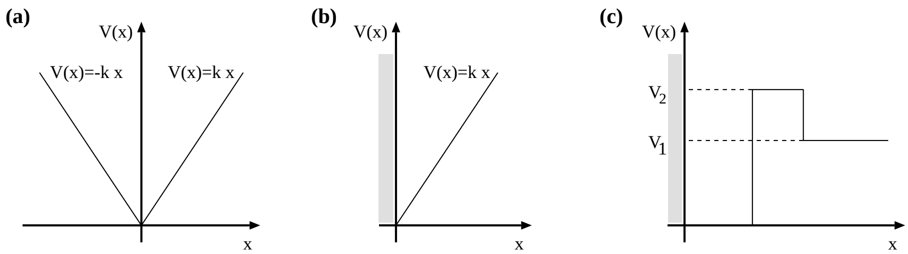
To be used: Mathematics Handbook, Physics Handbook and pocket calculator.

5 problems in 6 hours.
Each problem gives maximum 5 points.
Read the problems through, before you start solving them.
Please start each solution on a new paper.

1. a) Derive the continuity equation for a particle with mass m in a real potential $V(\vec{r})$. How is the equation interpreted?

b) Prove that two eigenstates of a hermitian operator with different eigenvalues are orthogonal.

2. **Part 1.** Consider the one-dimensional potentials (a), (b) and (c) below (the shaded regions are impenetrable walls).



a) Which potentials are invariant under parity ($x \to -x$)?

b) For each potential, state if it will have discrete or continuous energy spectrum (or both). Find the threshold energy above which the continuum will start.

c) Show that all energy eigenvalues of (b) are also energy eigenvalues of (a).

Part 2. Consider the following three potentials in three dimensions:

$$
\begin{aligned}
(a) \qquad V_1(\vec{r}) &= A(x^4 + y^4) + Bz^2 \\
(b) \qquad V_2(\vec{r}) &= e^{ax^2} e^{ay^2} e^{az^2} \\
(c) \qquad V_3(\vec{r}) &= C(2 + \cos\theta) r^2
\end{aligned}
$$

where x, y, z are Cartesian coordinates and r, θ, ϕ are spherical coordinates. A, B, C and a are constants.

a) For which of the above potentials is the time-independent Schrödinger equation separable in Cartesian coordinates?

b) For which potentials is it separable in spherical coordinates?

c) Which potentials are invariant under parity ($\vec{r} \to -\vec{r}$)?

d) For which potentials does the orbital angular momentum of a particle $\vec{L}$ commute with its Hamiltonian H?

3. We have a particle with mass m in the one-dimensional potential

$$V(x) = \begin{cases} \infty & , \quad x < -a \\ 0 & , \quad -a \leq x \leq a \\ \infty & , \quad x > a \end{cases} .$$

The wave function of the particle is given by

$$\psi(x) = \sqrt{\frac{2}{3}}\psi_1(x) + \sqrt{\frac{1}{3}}\psi_3(x)$$

where

$$\psi_n(x) \qquad ; \qquad n = 1, 2, 3, \ldots$$

are real normalized energy eigenstates (ordered with increasing energy) to the potential $V(x)$.

a) Write down $\psi_1(x)$ and $\psi_3(x)$.

b) What is the parity of $\psi(x)$?

c) If we measure the energy of the particle, what energies can we get and with what probabilities?

d) Determine $< x >$, $< p >$ and $< E >$.

e) Determine $< x^2 >$ and $< p^2 >$.

f) Determine $\Delta x \Delta p$.

Hint:

$$\int_0^{\pi/2} y^2 \cos ny \cos my \, dy = 2\frac{(-1)^{(n/2-m/2)}\pi mn}{(n^2 - m^2)^2}$$

where $n \neq m$ and n, m odd integers

and

$$\int_0^{\pi/2} y^2 \cos^2 ny \, dy = \frac{\pi}{48}\frac{\pi^2 n^2 - 6}{n^2} \qquad ; \qquad n \text{ odd integer}$$

4. An electron (neglect spin) of mass μ in a Coulomb potential $V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$ is in the energy eigenstate $\psi_{nlm} = \psi_{210}$.

a) How many degenerate states exist for this eigenvalue of the energy, E_2?

b) What is the minimum energy required to excite this particle in a state of orbital angular momentum $l = 3$?

c) Another electron is in the state $\psi = A\psi_{100} + B\psi_{322}$ (A, B constants). We know that the expectation value of L_z for this particle is less than $\hbar$. Show that the expectation value of the energy of this particle is less than E_2.

d) A magnetic field B is switched on in the z-direction, and has the effect of shifting the energy of all eigenstates of L_z by $\Delta E = -\frac{eBL_z}{2\mu}$. How strong must B be so that the ground state becomes an $l = 1$ state?

5. The wavefunction $\psi(x,t)$ is normalized and satisfies the one-dimensional Schrödinger equation for a free particle of mass m. Another wavefunction φ is given by

$$\varphi(x,t) = e^{iax+ibt}\,\psi(x - vt, t)$$

where a, b, v are constants.

a) Is φ normalized?

b) Show that $< \psi|p|\psi >= p_o$ and that $< \psi|x|\psi >= \frac{p_o}{m}t + x_o$ where p_o, x_o are constants. (Hint: Use Ehrenfest's theorem.)

c) Find a and b so that $\varphi(x,t)$ also satisfies the free Schrödinger's equation.

d) For the above values of a, b, find the expectation values $< \varphi|p|\varphi >$ and $< \varphi|x|\varphi >$ in terms of x_o, p_o and v.

e) If $< \psi|\frac{p^2}{2m}|\psi >= E_o$, find $< \varphi|\frac{p^2}{2m}|\varphi >$ in terms of E_o, x_o, p_o and v.

GOOD LUCK!