Solutions to selected problems in "Problems for Quantum Mechanics"

Joakim Edsjö

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Chapter I

Exercise I.1

Blackbody radiation: Planck's distribution law reads

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

We have measurements at

$$\lambda_0 = 6659\text{Å}$$

At $T_1 = 1336$ K we have $\rho(\lambda_0, T_1)$. Seek T_2 such that

$$\frac{\rho(\lambda_0, T_2)}{\rho(\lambda_0, T_1)} = 50$$

This yields

$$\frac{\exp\left(\frac{hc}{\lambda_0 k_B T_1}\right) - 1}{\exp\left(\frac{hc}{\lambda_0 k_B T_2}\right) - 1} = 50$$

$$\Rightarrow \exp\left(\frac{hc}{\lambda_0 k_B T_1}\right) - 1 = 50 \exp\left(\frac{hc}{\lambda_0 k_B T_2}\right) - 50$$

$$\Rightarrow \exp\left(\frac{hc}{\lambda_0 k_B T_2}\right) = \frac{1}{50} \exp\left(\frac{hc}{\lambda_0 k_B T_1}\right) + \frac{49}{50}$$

$$\Rightarrow \frac{hc}{\lambda_0 k_B T_2} = \ln\left(\frac{\exp\left(\frac{hc}{\lambda_0 k_B T_1}\right) + 49}{50}\right)$$

$$\Rightarrow T_2 = \frac{hc}{\lambda_0 k_B} \ln\left(\frac{\exp\left(\frac{hc}{\lambda_0 k_B T_1}\right) + 49}{50}\right) \approx 1764 \text{ K}$$

Exercise I.2

We have emission maximum at

$$\lambda_1 = 2.0\ \mu\text{m}.$$

The metal is heated and at the new emission maximum λ_2, we have

$$\frac{\rho(\lambda_2, T_2) d\lambda}{\rho(\lambda_1, T_1) d\lambda} = 1.50$$

Planck's distribution law reads

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

Find the wavelength at which $\rho(\lambda, T)$ has a maximum:

$$\begin{aligned} \frac{d\rho}{d\lambda} &= -\frac{8\pi hc}{\lambda^6} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1} \\ &\quad -\frac{8\pi hc}{\lambda^5} \frac{1}{\left(\exp\left(\frac{hc}{\lambda k_B T}\right) - 1\right)^2} \left(-\frac{hc}{\lambda^2 k_B T} \exp\left(\frac{hc}{\lambda k_B T}\right)\right) \\ &= \frac{8\pi hc}{\lambda^6 \left(\exp\left(\frac{hc}{\lambda k_B T}\right) - 1\right)} \left[\frac{hc}{\lambda k_B T} \frac{\exp\left(\frac{hc}{\lambda k_B T}\right)}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1} - 5\right] \end{aligned}$$

We now see that $\frac{d\rho}{d\lambda} = 0$ when

$$\frac{hc}{\lambda k_B T} \frac{1}{1 - \exp\left(\frac{hc}{\lambda k_B T}\right)} - 5 = 0$$

To solve this equation, let

$$x = \frac{hc}{\lambda k_B T}.$$

This yields

$$x = 5\left(1 - e^{-x}\right)$$

We see that x is slightly below 5. Hence let

$$x = 5 - \varepsilon.$$

This gives

$$5 - \varepsilon = 5 - 5e^{-5} e^{\varepsilon}$$

$$\Rightarrow \varepsilon = 5e^{-5}e^{\varepsilon}$$

Iteration gives

$$\varepsilon \approx 0.03489$$

Thus

$$\frac{hc}{\lambda k_B T} = 5 - 0.03489 = 4.965$$

$$\Rightarrow \lambda_{max} T = b \qquad ; \qquad b = \frac{hc}{4.965 k_B}$$

Now put $\lambda = \frac{b}{T}$ into Planck's distribution law:

$$\rho(\lambda_{max}, T) = \frac{8\pi hc T^5}{b^5} \frac{1}{\exp\left(\frac{hc}{k_B b}\right) - 1}$$

Thus we now have

$$\frac{\rho(\lambda_{max}, T_2)}{\rho(\lambda_{max}, T_1)} = 1.5$$

$$\Rightarrow T_2 = (1.5)^{1/5} T_1$$

where T_1 is given by

$$\lambda_1 T_1 = b$$

$$\Rightarrow T_1 = \frac{b}{\lambda_1} = \frac{hc}{4.965 k_B \lambda_1} \approx 1449 \text{ K}$$

We then finally get

$$T_2 = 1571 \text{ K}$$

and

$$\lambda_2 = \frac{b}{T_2} = \frac{hc}{4.965 k_B T_2} = 1.84 \ \mu\text{m}$$

Exercise I.3

a) The solar constant is given by

$$L = 1340 \text{ W/m}^2$$

and hence the mean value per square meter of the surface of the Earth is given by

$$\bar{L} = \frac{\pi R^2}{4\pi R^2} L = \frac{1}{4} L = 335 \text{ W/m}^2$$

b) The emissive power is given by

$$R = \frac{c\rho}{4}$$

and we further have

$$\rho_{tot} = aT^4 \qquad ; \qquad a = \frac{8\pi^5}{15} \frac{k_B^4}{h^3 c^3}$$

$$\Rightarrow R_{tot} = \frac{2\pi^5}{15} \frac{k_B^4}{h^3 c^2} T^4$$

Now assume that the total emissive power is equal to the mean solar constant, i.e. that the Earth emits energy at the same rate as it receives and that it is a black body. We then have

$$R_{tot} = \bar{L}$$

$$\Rightarrow \frac{2\pi^5}{15} \frac{k_B^4}{h^3 c^2} T^4 = \bar{L}$$

$$\Rightarrow T = 277.2 \text{ K} = 4.1^\circ\text{C}$$

c) The emission has its maximum at λ_{max} where

$$\lambda_{max} = \frac{hc}{4.965 k_B T} = 10.4 \ \mu\text{m}$$

Exercise I.4

In the first case we have

$$E_1 = h\nu_1 = \frac{hc}{\lambda_1} \qquad ; \qquad \lambda_1 = 4358 \text{ Å}$$

and in the second case

$$E_2 = h\nu_2 = \frac{hc}{\lambda_2} \qquad ; \qquad \lambda_2 = 2537 \text{ Å}$$

To stop the electrons in case 2 we need a voltage $V_0 = 2.04$ V, i.e. the most energetic electrons have the energy

$$E_e^{max} = V_0 e$$

which is equal to the difference between E_2 and E_1, i.e.

$$E_2 - E_1 = E_e^{max}$$

$$\Rightarrow hc\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2}\right) = V_0 e$$

$$\Rightarrow h = \frac{V_0 e}{c\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2}\right)} \simeq 6.62 \times 10^{-34} \text{ Js}$$

Chapter II

Exercise II.1

We have the wave function

$$\psi(x,0) = \begin{cases} \frac{1}{\sqrt{2L}} e^{ip_0 x/\hbar} & ; \quad |x| < L \\ 0 & ; \quad |x| > L \end{cases}$$

and we want to calculate the momentum wave function $\phi(p,0)$,

$$\phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ipx/\hbar} dx$$

If we insert the configuration space wave function into this we get

$$\begin{aligned} \phi(p,0) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-L}^{L} \frac{1}{\sqrt{2L}} e^{ip_0 x/\hbar} e^{-ipx/\hbar} dx \\ &= \frac{1}{2\sqrt{\pi L\hbar}} \int_{-L}^{L} e^{i(p_0-p)x/\hbar} dx \\ &= \frac{1}{2\sqrt{\pi L\hbar}} \left[\frac{\hbar}{i(p_0-p)} e^{i(p_0-p)x/\hbar}\right]_{-L}^{L} \\ &= \frac{\hbar}{2\sqrt{\pi L\hbar}} \frac{1}{i(p_0-p)} \left(e^{i(p_0-p)x/\hbar} - e^{-i(p_0-p)x/\hbar}\right) \\ &= \sqrt{\frac{\hbar}{L\pi}} \frac{1}{2i(p_0-p)} 2i \sin\left(\frac{(p_0-p)L}{\hbar}\right) \\ &= \sqrt{\frac{\hbar}{L\pi}} \frac{\sin\left(\frac{(p_0-p)L}{\hbar}\right)}{p_0-p} \end{aligned}$$

Now assume that the particle is free, $\phi(p,t)$ is then given by

$$\phi(p,t) = \phi(p,0) e^{-iE(p)t/\hbar} = \phi(p,0) e^{-ip^2 t/2m\hbar}$$

Finally, $\psi(x,t)$ is given by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p,t) e^{ipx/\hbar} dp$$

Exercise II.2

At $t = 0$ we have the wave function

$$\psi(x,0) = \frac{1}{\sqrt{\Delta_{0x}}} \frac{1}{(2\pi)^{1/4}} e^{-\frac{x^2}{4\Delta_{0x}^2} + \frac{i}{\hbar} p_0 x}$$

and we want to calculate the momentum wave function $\phi(p,0)$,

$$\begin{aligned}
\phi(p,0) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x,0)e^{-\frac{ipx}{\hbar}}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{(2\pi\Delta_{0x}^2)^{1/4}}e^{-\frac{x^2}{4\Delta_{0x}^2}}e^{\frac{i(p_0-p)x}{\hbar}}dx \qquad (1)
\end{aligned}$$

According to the hint, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{2a}{\pi\hbar^2}\right)^{1/4}\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i(p-p_0)x}{\hbar}}e^{-\frac{a(p-p_0)^2}{\hbar^2}}dp = \frac{1}{(2\pi a)^{1/4}}e^{-\frac{x^2}{4a}} \\
&\Re(a) > 0 \qquad (2)
\end{aligned}$$

If we now let $a = \frac{\hbar^2}{4\Delta_{0x}^2}$ and substitute $p - p_0 \to x$ and $x \to p0 - p$ in the hint, Eq. (2), we use the hint to calculate the momentum wave function, Eq. (1)

$$\begin{aligned}
\phi(p,0) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2a}{\pi\hbar^2}\right)e^{-\frac{ax^2}{\hbar^2}}e^{\frac{i(p_0-p)x}{\hbar}}dx \\
&= \frac{1}{(2\pi a)^{1/4}}e^{-\frac{(p-p_0)^2}{4a}} \\
&= \left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}e^{-\frac{\Delta_{0x}^2(p-p_0)^2}{\hbar^2}}
\end{aligned}$$

Since the particle is free, $\phi(p,t)$ is given by

$$\begin{aligned}
\phi(p,t) &= \phi(p,0)e^{-\frac{iE(p)t}{\hbar}} = \phi(p,0)e^{-\frac{ip^2t}{2m\hbar}} \\
&= \left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}e^{-\frac{\Delta_{0x}^2(p-p_0)^2}{\hbar^2}}e^{-\frac{ip^2t}{2m\hbar}}
\end{aligned}$$

The configuration space wave function $\psi(x,t)$ is then given by

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p,t)e^{\frac{ipx}{\hbar}}dp \\
&= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\Delta_{0x}^2(p-p_0)^2}{\hbar^2}}e^{-\frac{ip^2t}{2m\hbar}}e^{\frac{ipx}{\hbar}}dp \\
&= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}\int_{-\infty}^{\infty}e^{-(p-p_0)^2\left[\frac{\Delta_{0x}^2}{\hbar^2}+\frac{it}{2m\hbar}\right]}e^{\frac{2ipp_0t}{2m\hbar}}e^{\frac{itp_0^2}{2m\hbar}}e^{\frac{ipx}{\hbar}}dp \\
&= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}e^{\frac{ip_0^2t}{2m\hbar}}\int_{-\infty}^{\infty}e^{-(p-p_0)^2\left[\frac{\Delta_{0x}^2}{\hbar^2}+\frac{it}{2m\hbar}\right]}e^{ip\left[\frac{x}{\hbar}-\frac{p_0t}{m\hbar}\right]}dp \\
&= \left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}e^{\frac{ip_0x}{\hbar}}e^{-\frac{ip_0^2t}{2m\hbar}}\frac{1}{\sqrt{2\pi\hbar}}\times \\
&\quad \int_{-\infty}^{\infty}e^{-(p-p_0)^2\left[\frac{\Delta_{0x}^2}{\hbar^2}+\frac{it}{2m\hbar}\right]}e^{i(p-p_0)\frac{x-p_0t}{\hbar}}dp \qquad (3)
\end{aligned}$$

With $a = \Delta_{0x}^2 + \frac{i\hbar t}{2m}$ and the hint, Eq. (2), we get

$$\begin{aligned}
\psi(x,t) &= \left(\frac{2\Delta_{0x}^2}{\pi\hbar^2}\right)^{1/4}e^{\frac{ip_0x}{\hbar}}e^{-\frac{ip_0^2t}{2m\hbar}}\left(\frac{\pi\hbar^2}{2a}\right)^{1/4}\frac{1}{(2\pi a)^{1/4}}e^{-\frac{(x-p_0t/m)^2}{4a}} \\
&= \left(\frac{\Delta_{0x}^2}{2\pi a^2}\right)^{1/4}e^{\frac{ip_0x}{\hbar}}e^{-\frac{ip_0^2t}{2m\hbar}}e^{-\frac{(x-p_0t/m)^2}{4a}} \\
&= \frac{1}{(2\pi)^{1/4}}\sqrt{\frac{\Delta_{0x}}{\Delta_{0x}^2+i\hbar t/2m}}e^{-\frac{ip_0^2t}{2m\hbar}}e^{\frac{ip_0x}{\hbar}}e^{-\frac{1}{4}\frac{(x-p_0t/m)^2}{\Delta_{0x}^2+i\hbar t/2m}}
\end{aligned}$$

Hence, the probability density is given by

$$\begin{aligned}
|\psi(x,t)|^2 &= \frac{1}{\sqrt{2\pi}}\Delta_{0x}\sqrt{\frac{1}{\Delta_{0x}^2-i\hbar t/2m}\frac{1}{\Delta_{0x}^2+i\hbar t/2m}}\times \\
&\quad e^{-\frac{1}{4}\left(x-\frac{p_0t}{m}\right)^2\left[\frac{1}{\Delta_{0x}^2-i\hbar t/2m}+\frac{1}{\Delta_{0x}^2+i\hbar t/2m}\right]} \\
&= \frac{1}{\sqrt{2\pi}}\Delta_{0x}\sqrt{\frac{1}{\Delta_{0x}^4+\frac{\hbar^2t^2}{4m^2}}}e^{-\frac{1}{4}\left(x-\frac{p_0t}{m}\right)^2\frac{2\Delta_{0x}^2}{\Delta_{0x}^4+\frac{\hbar^2t^2}{4m^2}}}
\end{aligned}$$

Now introduce

$$\Delta_x = \sqrt{\Delta_{0x}^2+\frac{\hbar^2t^2}{4m^2\Delta_{0x}^2}} = \Delta_{0x}\sqrt{1+\frac{\hbar^2t^2}{4m^2\Delta_{0x}^4}}$$

which yields

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\frac{1}{\Delta_x}e^{-\frac{\left(x-\frac{p_0t}{m}\right)^2}{2\Delta_x^2}}$$

i e the center of the wave packet moves like a classical particle, $x = p_0t/m$, and the width, Δ_x, increases with time.

Exercise II.3

The commutator between two operators $\hat{A}$ and $\hat{B}$ is defined as

$$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

a) We now want to prove

$$\left[\hat{A}\hat{C},\hat{B}\right] = \hat{A}\left[\hat{C},\hat{B}\right] + \left[\hat{A},\hat{B}\right]\hat{C} \qquad (1)$$

The left hand side of Eq. (1) can be written

$$\begin{aligned}
\left[\hat{A}\hat{C},\hat{B}\right] &= \hat{A}\hat{C}\hat{B}-\hat{B}\hat{A}\hat{C}=\hat{A}\hat{C}\hat{B}-\hat{A}\hat{B}\hat{C}+\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}\\
&= \hat{A}\Big(\underbrace{\hat{C}\hat{B}-\hat{B}\hat{C}}_{[\hat{C},\hat{B}]}\Big)+\Big(\underbrace{\hat{A}\hat{B}-\hat{B}\hat{A}}_{[\hat{A},\hat{B}]}\Big)\hat{C}\\
&= \hat{A}\left[\hat{C},\hat{B}\right]+\left[\hat{A},\hat{B}\right]\hat{C}
\end{aligned}$$

Q.E.D.

b) We now want to prove

$$\left[\hat{A},\hat{C}\hat{B}\right]=\hat{C}\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]\hat{B} \tag{2}$$

The left hand side of Eq. (2) can be written

$$\begin{aligned}
\left[\hat{A},\hat{C}\hat{B}\right] &= \hat{A}\hat{C}\hat{B}-\hat{C}\hat{B}\hat{A}=\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}+\hat{C}\hat{A}\hat{B}-\hat{C}\hat{B}\hat{A}\\
&= \Big(\underbrace{\hat{A}\hat{C}-\hat{C}\hat{A}}_{[\hat{A},\hat{C}]}\Big)\hat{B}+\hat{C}\Big(\underbrace{\hat{A}\hat{B}-\hat{B}\hat{A}}_{[\hat{A},\hat{B}]}\Big)\\
&= \hat{C}\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]\hat{B}
\end{aligned}$$

Q.E.D.

Exercise II.4

a) We want to prove the cyclic identity

$$\left[\hat{A},\left[\hat{B},\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]=0 \tag{1}$$

Consider the left-hand side of Eq. (1)

$$\begin{aligned}
L.H. &= \left[\hat{A},\left[\hat{B},\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]\\
&= \hat{A}\left[\hat{B},\hat{C}\right]-\left[\hat{B},\hat{C}\right]\hat{A}+\hat{B}\left[\hat{C},\hat{A}\right]-\left[\hat{C},\hat{A}\right]\hat{B}+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]\\
&= \underbrace{-\hat{A}\left[\hat{C},\hat{B}\right]-\left[\hat{C},\hat{A}\right]\hat{B}}_{-[\hat{C},\hat{A}\hat{B}]}+\underbrace{\left[\hat{C},\hat{B}\right]\hat{A}+\hat{B}\left[\hat{C},\hat{A}\right]}_{[\hat{C},\hat{B}\hat{A}]}+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]\\
&= -\Big[\hat{C},\underbrace{\hat{A}\hat{B}-\hat{B}\hat{A}}_{[\hat{A},\hat{B}]}\Big]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]\\
&= -\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]=0
\end{aligned}$$

Q.E.D.

b) Prove that

$$\left[V(x),\frac{d}{dx}\right]=-\frac{dV}{dx} \tag{2}$$

Apply the left-hand side of Eq. (2) to an arbitrary wave function $\psi(x)$

$$\begin{aligned}
\left[V(x),\frac{d}{dx}\right]\psi(x) &= \left(V(x)\frac{d}{dx}-\frac{d}{dx}V(x)\right)\psi(x)\\
&= V(x)\frac{d\psi}{dx}-\frac{d}{dx}\left(V(x)\psi(x)\right)\\
&= V(x)\frac{d\psi}{dx}-\frac{dV}{dx}\psi(x)-V(x)\frac{d\psi}{dx}\\
&= -\frac{dV}{dx}\psi(x)
\end{aligned}$$

Since $\psi(x)$ is arbitrary we have

$$\left[V(x),\frac{d}{dx}\right]=-\frac{dV}{dx}$$

Q.E.D.

Exercise II.6

a) Using that $\hat{L}_x=\hat{y}\hat{p}_z-\hat{z}\hat{p}_y$ we get

$$\begin{aligned}
&\left[\hat{L}_x,\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2\right]=\left[\hat{y}\hat{p}_z-\hat{z}\hat{p}_y,\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2\right]=\{[\hat{p}_i,\hat{p}_j]=0\}\\
&= \left[\hat{y},\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2\right]\hat{p}_z-\left[\hat{z},\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2\right]\hat{p}_y\\
&= \underbrace{\left[\hat{y},\hat{p}_x^2\right]}_{0}\hat{p}_z+\left[\hat{y},\hat{p}_y^2\right]\hat{p}_z+\underbrace{\left[\hat{y},\hat{p}_z^2\right]}_{0}\hat{p}_z-\underbrace{\left[\hat{z},\hat{p}_x^2\right]}_{0}\hat{p}_y-\underbrace{\left[\hat{z},\hat{p}_y^2\right]}_{0}\hat{p}_y-\left[\hat{z},\hat{p}_z^2\right]\hat{p}_y\\
&= \underbrace{[\hat{y},\hat{p}_y]}_{i\hbar}\hat{p}_y\hat{p}_z+\hat{p}_y\underbrace{[\hat{y},\hat{p}_y]}_{i\hbar}\hat{p}_z-\underbrace{[\hat{z},\hat{p}_z]}_{i\hbar}\hat{p}_z\hat{p}_y-\hat{p}_z\underbrace{[\hat{z},\hat{p}_z]}_{i\hbar}\hat{p}_y\\
&= i\hbar\left(\hat{p}_y\hat{p}_z+\hat{p}_y\hat{p}_z-\hat{p}_z\hat{p}_y-\hat{p}_z\hat{p}_y\right)=2i\hbar\left[\hat{p}_y,\hat{p}_z\right]=0
\end{aligned}$$

b) In the same way we get

$$\begin{aligned}
&\left[\hat{L}_x,\hat{x}^2+\hat{y}^2+\hat{z}^2\right]=\left[\hat{y}\hat{p}_z-\hat{z}\hat{p}_y,\hat{x}^2+\hat{y}^2+\hat{z}^2\right]=\{[\hat{x}_i,\hat{x}_j]=0\}\\
&= \hat{y}\left[\hat{p}_z,\hat{x}^2+\hat{y}^2+\hat{z}^2\right]-\hat{z}\left[\hat{p}_y,\hat{x}^2+\hat{y}^2+\hat{z}^2\right]\\
&= \hat{y}\underbrace{\left[\hat{p}_z,\hat{x}^2\right]}_{0}+\hat{y}\underbrace{\left[\hat{p}_z,\hat{y}^2\right]}_{0}+\hat{y}\left[\hat{p}_z,\hat{z}^2\right]-\hat{z}\underbrace{\left[\hat{p}_y,\hat{x}^2\right]}_{0}-\hat{z}\left[\hat{p}_y,\hat{x}^2\right]-\hat{z}\underbrace{\left[\hat{p}_y,\hat{z}^2\right]}_{0}\\
&= \hat{y}\hat{z}\underbrace{[\hat{p}_z,\hat{z}]}_{-i\hbar}+\hat{y}\underbrace{[\hat{p}_z,\hat{z}]}_{-i\hbar}\hat{z}-\hat{z}\hat{y}\underbrace{[\hat{p}_y,\hat{y}]}_{-i\hbar}-\hat{z}\underbrace{[\hat{p}_y,\hat{y}]}_{-i\hbar}\hat{y}\\
&= 2i\hbar\left(\hat{z}\hat{y}-\hat{y}\hat{z}\right)=2i\hbar\left[\hat{z},\hat{y}\right]=0
\end{aligned}$$

c) Let the commutator $\left[\hat{L}_x, V(r)\right]$ act on an arbitrary wave function $\Psi(x)$,

$$\begin{aligned}
\left[\hat{L}_x, V(r)\right]\Psi(x) &= \left[\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, V(r)\right]\Psi(x) \\
&= \left[-i\hbar\hat{y}\frac{\partial}{\partial z} + i\hbar z\frac{\partial}{\partial y}, V(r)\right]\Psi(x) \\
&= i\hbar\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)V(r)\Psi(x) - V(r)i\hbar\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\Psi(x) \\
&= i\hbar\Psi(x)\left(z\frac{\partial V(r)}{\partial y} - y\frac{\partial V(r)}{\partial z}\right) + i\hbar V(r)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\Psi(x) \\
&\quad -i\hbar V(r)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\Psi(x) \\
&= i\hbar\left(z\frac{\partial V(r)}{\partial y} - y\frac{\partial V(r)}{\partial z}\right)\Psi(x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left[\hat{L}_x, V(r)\right] &= i\hbar\left(z\frac{\partial V(r)}{\partial y} - y\frac{\partial V(r)}{\partial z}\right) \\
&= i\hbar\left(z\frac{\partial V}{\partial r}\frac{\partial r}{\partial y} - y\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}\right)
\end{aligned}$$

Now use that

$$r = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad \begin{cases} \frac{\partial r}{\partial y} = \frac{2y}{2r} = \frac{y}{r} \\ \frac{\partial r}{\partial z} = \frac{2z}{2r} = \frac{z}{r} \end{cases}$$

which finally gives

$$\left[\hat{L}_x, V(r)\right] = i\hbar\left(z\frac{y}{r}\frac{\partial V}{\partial r} - y\frac{z}{r}\frac{\partial V}{\partial r}\right) = 0$$

Extra. Consider motion in a central field, i e with the Hamiltonian

$$\hat{H} = \frac{\hat{\mathrm{p}}^2}{2m} + V(r) \quad ; \quad \hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

Due to symmetry reasons, the commutation relations proven above are valid for $\hat{L}_y$ and $\hat{L}_z$ as well, i e

$$\left[\hat{\mathbf{L}}, \hat{\mathrm{p}}^2\right] = \left[\hat{\mathbf{L}}, \hat{\mathrm{x}}^2\right] = \left[\hat{\mathbf{L}}, V(r)\right] = 0.$$

Now consider

$$\begin{aligned}
\frac{d}{dt}\langle\hat{\mathbf{L}}\rangle &= \frac{d}{dt}\langle\Psi|\hat{\mathbf{L}}|\Psi\rangle \\
&= \langle\underbrace{\frac{\partial\Psi}{\partial t}}_{\frac{1}{i\hbar}\hat{H}\Psi}|\hat{\mathbf{L}}|\Psi\rangle + \langle\Psi|\underbrace{\frac{\partial\hat{\mathbf{L}}}{\partial t}}_{\text{0 since operators are time independent}}|\Psi\rangle + \langle\Psi|\hat{\mathbf{L}}|\underbrace{\frac{\partial\Psi}{\partial t}}_{\frac{1}{i\hbar}\hat{H}\Psi}\rangle \\
&= -\frac{1}{i\hbar}\langle\hat{H}\Psi|\hat{\mathbf{L}}|\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{\mathbf{L}}|\hat{H}\Psi\rangle = \left\{\hat{H}\ \text{Hermitian}\right\} \\
&= \frac{1}{i\hbar}\langle\Psi|\hat{\mathbf{L}}\hat{H} - \hat{H}\hat{\mathbf{L}}|\Psi\rangle = \frac{1}{i\hbar}\langle\underbrace{\left[\hat{\mathbf{L}}, \hat{H}\right]}_{\text{0 in this case}}\rangle = 0
\end{aligned}$$

which means that $\langle\hat{\mathbf{L}}\rangle$ is a constant of motion. Note that all observables that commute with $\hat{H}$ are constants of motion.

Exercise II.7

The operator $\hat{A}$ has n eigenvalues (non-degenerate) $a_1, a_2, \ldots, a_n$ with eigenfunctions $u_1, u_2, \ldots, u_n$. Consider the operator

$$f(\hat{A}) = (\hat{A} - a_1)(\hat{A} - a_2)\cdots(\hat{A} - a_n) = (\hat{A} - a_k)g_k(\hat{A}).$$

An arbitrary wave function Ψ can be written

$$\Psi = \sum_{m=1}^{n} c_m u_m$$

First consider

$$\begin{aligned}
f(\hat{A})\Psi &= \prod_{l=1}^{n}(\hat{A} - a_l)\sum_{m=1}^{n} c_m u_m = \sum_{m=1}^{n} c_m \prod_{l=1}^{n}(\hat{A} - a_l)u_m \\
&= \sum_{m=1}^{n} c_m \underbrace{\prod_{l=1}^{n}(a_m - a_l)}_{0} u_m = 0
\end{aligned}$$

and then

$$\begin{aligned}
\frac{g_k(\hat{A})}{g_k(a_k)}\Psi &= \frac{1}{g_k(a_k)}\prod_{l=1, l\neq k}^{n}(\hat{A} - a_l)\sum_{m=1}^{n} c_m u_m \\
&= \frac{1}{g_k(a_k)}\sum_{m=1}^{n} c_m \prod_{l=1, l\neq k}^{n}(\hat{A} - a_l)u_m
\end{aligned}$$

$$= \frac{1}{g_k(a_k)} \sum_{m=1}^{n} c_m \underbrace{\prod_{l=1, l\neq k}^{n} (a_m - a_l)}_{0 \text{ for all } m \neq k} u_m$$

$$= \frac{1}{g_k(a_k)} c_k \underbrace{\prod_{l=1, l\neq k}^{n} (a_k - a_l)}_{g_k(a_k)} u_k = c_k u_k$$

i e $g_k(\hat{A})/g_k(a_k)$ projects out the u_k part of Ψ.

Chapter III

Exercise III.1

a) At $t = 0$ we have

$$\Psi(\mathbf{r}, 0) = ae^{i\mathbf{p}_1\cdot\mathbf{r}/\hbar} + be^{i\mathbf{p}_2\cdot\mathbf{r}/\hbar} \quad ; \quad a, b \in \Re.$$

We see that Ψ is a superposition of Ψ_1 and Ψ_2 where Ψ_1 and Ψ_2 are plane waves,

$$\begin{cases} \Psi_1(\mathbf{r}, t) &= e^{\frac{i}{\hbar}(\mathbf{p}_1\cdot\mathbf{r} - E_1 t)} = e^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} \\ \Psi_2(\mathbf{r}, t) &= e^{\frac{i}{\hbar}(\mathbf{p}_2\cdot\mathbf{r} - E_2 t)} = e^{\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)} \end{cases}$$

$\Psi_1(\mathbf{r}, t)$ and $\Psi_2(\mathbf{r}, t)$ are solutions of the time dependent Schrödinger equation. This means that $\Psi(\mathbf{r}, t)$ is given by

$$\Psi(\mathbf{r}, t) = a\Psi_1(\mathbf{r}, t) + b\Psi_2(\mathbf{r}, t) = ae^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} + be^{\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)}$$

where $\Psi(\mathbf{r}, t)$ is also a solution of the time dependent Schrödinger equation (since the equation is linear).

b) The probability current is given by

$$\mathbf{S}(\mathbf{r}, t) = \Re\left[\Psi^*(\mathbf{r}, t)\frac{\hbar}{im}\nabla\Psi(\mathbf{r}, t)\right].$$

Noting that

$$\nabla e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} = \nabla e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z)} = \frac{i}{\hbar}\left(p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}\right)e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} = \frac{i}{\hbar}\mathbf{p}e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}}$$

we get

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) = \Re\Bigg\{ &\left[ae^{-\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} + be^{-\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)}\right]\frac{\hbar}{im} \\ &\left[a\frac{i}{\hbar}\mathbf{p}_1 e^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} + b\frac{i}{\hbar}\mathbf{p}_2 e^{\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)}\right]\Bigg\} \\ = \ &\Re\left[\frac{1}{m}a^2\mathbf{p}_1 + \frac{1}{m}b^2\mathbf{p}_2 + \frac{1}{m}ab\mathbf{p}_1 e^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t - \mathbf{p}_2\cdot\mathbf{r} + \frac{\mathbf{p}_2^2}{2m}t\right)}\right. \\ &\left. + \frac{1}{m}ab\mathbf{p}_2 e^{\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t - \mathbf{p}_1\cdot\mathbf{r} + \frac{\mathbf{p}_1^2}{2m}t\right)}\right]. \end{aligned}$$

Since

$$\begin{cases} \Re\left(e^{ic}\right) &= \Re\left[\cos c + i\sin c\right] = \cos c \\ \Re\left(e^{-ic}\right) &= \Re\left[\cos c - i\sin c\right] = \cos c \end{cases} \quad ; \quad c \in \Re$$

we finally get

$$\begin{aligned} \mathbf{S}(\mathbf{r},t) &= \frac{a^2}{m}\mathbf{p}_1 + \frac{b^2}{m}\mathbf{p}_2 + \frac{ab}{m}\mathbf{p}_1 \cos\left[\frac{1}{\hbar}\left((\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{r} - \frac{\mathbf{p}_1^2 - \mathbf{p}_2^2}{2m}t\right)\right] \\ &\quad + \frac{ab}{m}\mathbf{p}_2 \cos\left[\frac{1}{\hbar}\left((\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{r} - \frac{\mathbf{p}_1^2 - \mathbf{p}_2^2}{2m}t\right)\right] \\ &= \frac{1}{m}\left\{a^2\mathbf{p}_1 + b^2\mathbf{p}_2 + ab(\mathbf{p}_1 + \mathbf{p}_2)\cos\left[\frac{1}{\hbar}(\mathbf{p}_1 - \mathbf{p}_2)\cdot\left(\mathbf{r} - \frac{\mathbf{p}_1 + \mathbf{p}_2}{2m}t\right)\right]\right\}. \end{aligned}$$

c) The probability density is given by

$$\begin{aligned} \rho(\mathbf{r},t) &= |\Psi(\mathbf{r},t)|^2 = \Psi^*\Psi \\ &= \left[a e^{-\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} + b e^{-\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)}\right]\left[a e^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t\right)} + b e^{\frac{i}{\hbar}\left(\mathbf{p}_2\cdot\mathbf{r} - \frac{\mathbf{p}_2^2}{2m}t\right)}\right] \\ &= a^2 + b^2 + ab\left[e^{\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t - \mathbf{p}_2\cdot\mathbf{r} + \frac{\mathbf{p}_2^2}{2m}t\right)} + e^{-\frac{i}{\hbar}\left(\mathbf{p}_1\cdot\mathbf{r} - \frac{\mathbf{p}_1^2}{2m}t - \mathbf{p}_2\cdot\mathbf{r} + \frac{\mathbf{p}_2^2}{2m}t\right)}\right] \\ &= a^2 + b^2 + 2ab\cos\left[\frac{1}{\hbar}(\mathbf{p}_1 - \mathbf{p}_2)\cdot\left(\mathbf{r} - \frac{\mathbf{p}_1 + \mathbf{p}_2}{2m}t\right)\right] \end{aligned}$$