Analytical Mechanics

Summary

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Newtonian Mechanics I

Newtonian mechanics = vector mechanics

Newton's laws

I. "Law of inertia"

II. "Force law"

III. "Law of action and reaction"

In an inertial system we have

$$\frac{d}{dt}\vec{p}(t) = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$$

Coordinate systems

Cartes. coord.	Cylindrical coord.	Spher. coord.
(x, y, z)	(ρ, φ, z)	(r, θ, φ)
$\begin{cases} \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \\ \vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} \end{cases}$	$\begin{cases} \vec{r} = \rho\hat{\rho} + z\hat{z} \\ \vec{v} = \dots \\ \vec{a} = \dots \end{cases}$	$\begin{cases} \vec{r} = r\hat{r} \\ \vec{v} = \dots \\ \vec{a} = \dots \end{cases}$

Isolated 2-body system

$$\begin{cases} M\ddot{\vec{r}}_s = 0 \\ \mu\ddot{\vec{r}} = \vec{F} \end{cases}$$

Newtonian Mechanics II

Central forces

$$\vec{F} = f(r)\hat{r}$$
$$\Downarrow$$

$$\begin{cases} \vec{l} = \vec{r} \times \vec{p} = \text{const.} = \mu r^2 \dot{\varphi} \hat{z} \\ E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = \text{const.} \end{cases}$$
$$\Downarrow$$

Planar motion!

Kepler's laws (for $U(r) = \frac{A}{r}$)

I. $r(\varphi) = \frac{p}{1+\epsilon \cos \varphi}$

II. $\frac{1}{2} r^2 \dot{\varphi} = \text{const.}$

III. $\frac{a^3}{T^2} = \text{const.}$

Conservative force fields

$$\nabla \times \vec{F} = 0$$

if and only if $\vec{F}$ is a conservative force field.

Newtonian Mechanics III

Particle systems

$$m_i\ddot{\vec{r}}_i = \sum_{k\neq i}^{n} \vec{F}_{ki} + \vec{F}_i^{(e)}$$

(i) Law of the center of mass motion

$$M\ddot{\vec{r}}_s = \sum_{i}^{n} \vec{F}_i^{(e)}$$

(ii) Momentum law (torque law)

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \vec{l}_i\right) = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^{(e)}$$

(iii) Energy law

$$\frac{d}{dt}(T+U) = \sum_{i=1}^{n} \vec{v}_i \times \vec{F}_i^{(e)}$$

Especially for an isolated system

$$\begin{cases} \vec{P} &= \text{const.} \\ \vec{L} &= \text{const.} \\ E &= \text{const.} \end{cases}$$

Newtonian Mechanics IV

2-body problem with a central force

$$\begin{aligned} \varphi - \varphi_0 &= \pm l \int_{r_0}^{r(\varphi)} \frac{dr}{r^2\sqrt{2\mu(E - U_{\text{eff}}(r))}} \\ U_{\text{eff}}(r) &= U(r) + \frac{l^2}{2\mu r^2} \end{aligned}$$

If

- $r(t) \geq r_{\text{min}}$ we have scattering states.
  $\Rightarrow$ We can learn about the potential by studying the differential cross section,
  $$\frac{d\sigma}{d\omega} = \frac{b(\theta^*)}{\sin\theta^*}\left|\frac{db(\theta^*)}{d\theta^*}\right|$$
- $r_{\text{min}} \leq r(t) \leq r_{\text{max}}$ we have bound states. Symmetric around SA and SP where S is the force center, P is pericenter and A is apocenter.

Canonical Mechanics I

Consider n particles with masses m_i subject to Λ independent constraints

$$f_\lambda(\vec{r}_1, \ldots, \vec{r}_n, t) = 0 \qquad , \qquad \lambda = 1, \ldots, \Lambda$$

Generalized coordinates

A set of independent coordinates,

$$q_k(t) \quad , \quad k = 1, \ldots, f, \quad , \quad f = 3n - \Lambda$$

that the constraints into account.

Lagrange's equations

Define

$$L(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, t) = T(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) - U(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, t)$$

Then *Lagrange's equations* are fulfilled

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad , \qquad k = 1, \ldots, f}$$

Canonical Mechanics II

Hamilton's equations

Define the *Hamilton function*

$$H(\underset{\sim}{q}, \underset{\sim}{p}, t) = \sum_k \dot{q}_k p_k - L(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, t)$$

with the to q_k *canonically conjugated momentum* p_k given by

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Hamilton's canonical equations are then

$$\boxed{\frac{\partial H}{\partial p_k} = \dot{q}_k \quad ; \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k}$$

We have gone from a system of f second order differential equations (Lagrange) to $2f$ first order differential equations (Hamilton).

Canonical systems and transformations

Canonical system

A mechanical system is *canonical* if it can be described by a Hamilton function $H = H(\underset{\sim}{q}, \underset{\sim}{p}, t)$ such that Hamilton's equations are fulfilled.

Canonical transformations

A transformation

$$\begin{cases} \{\underset{\sim}{q}, \underset{\sim}{p}\} & \longrightarrow \quad \{\underset{\sim}{Q}, \underset{\sim}{P}\} \\ H(\underset{\sim}{q}, \underset{\sim}{p}, t) & \longrightarrow \quad K(\underset{\sim}{Q}, \underset{\sim}{P}, t) \end{cases}$$

is called canonical if it preserves the structure on the canonical equations, i.e. if

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \qquad \longrightarrow \qquad \begin{cases} \dot{Q}_i = \frac{\partial K}{\partial P_i} \\ \dot{P}_i = -\frac{\partial K}{\partial Q_i} \end{cases}$$

Remark. Both variables and the Hamilton function are transformed.

Canonical transformations

Class A. $F_1 = F_1(\underset{\sim}{q}, \underset{\sim}{Q}, t)$ - generating function

$$p_i = \frac{\partial F_1}{\partial q_i} \quad ; \quad P_j = -\frac{\partial F_1}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_1}{\partial t}$$

Class B. $F_2 = F_2(\underset{\sim}{q}, \underset{\sim}{P}, t)$ - generating function

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_j = \frac{\partial F_2}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_2}{\partial t}$$

Class C. $F_3 = F_3(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial F_3}{\partial p_i} \quad ; \quad P_j = -\frac{\partial F_3}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_3}{\partial t}$$

Class D. $F_4 = F_4(\underset{\sim}{P}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad ; \quad Q_j = \frac{\partial F_4}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_4}{\partial t}$$

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$$\begin{cases} -F_2 = (\mathcal{L}F_1)(\underset{\sim}{Q}) = \sum_k Q_k \frac{\partial F_1}{\partial Q_k} - F_1 \\ -F_3 = (\mathcal{L}F_1)(\underset{\sim}{q}) = \sum_k q_k \frac{\partial F_1}{\partial q_k} - F_1 \\ -F_4 = (\mathcal{L}F_1)(\underset{\sim}{q}, \underset{\sim}{Q}) = \sum_k \left[q_k \frac{\partial F_1}{\partial q_k} + Q_k \frac{\partial F_1}{\partial Q_k} \right] - F_1 \end{cases}$$

Hamilton-Jacobi's equations I

Hamilton-Jacobi's time dependent equation

$$\boxed{H(q_i, \frac{\partial S}{\partial q_i}, t) + \frac{\partial S}{\partial t} = 0}$$

gives *the action function* $S(\underset{\sim}{q}, \underset{\sim}{\alpha}, t)$ that with $\underset{\sim}{P} = \underset{\sim}{\alpha}$ generates the canonical transformation (class B)

$$\{\underset{\sim}{q}, \underset{\sim}{p}, H\} \longrightarrow \{\underset{\sim}{Q}, \underset{\sim}{P}, K = 0\}$$

Hamilton-Jacobi's time independent equation

If $\frac{\partial H}{\partial t} = 0$ we can write

$$S(\underset{\sim}{q}, \underset{\sim}{\alpha}, t) = W(\underset{\sim}{q}, \underset{\sim}{\alpha}) - E(\underset{\sim}{\alpha})t$$

$$\Rightarrow \quad \boxed{H(q_i, \frac{\partial W}{\partial q_i}) = E}$$

The canonical transformation can be derived from *the reduced action function* $W(\underset{\sim}{q}, \underset{\sim}{\alpha})$ in two ways:

1. Insert $W(\underset{\sim}{q}, \underset{\sim}{\alpha})$ in S and let S generate the canonical transformation.

2. Let $W(\underset{\sim}{q}, \underset{\sim}{\alpha})$ generate the canonical transformation directly.

Hamilton-Jacobi's equations II

Method 1 - transformation through $S(\underset{\sim}{q}, \underset{\sim}{\alpha}, t)$

$$p_i = \frac{\partial S}{\partial q_i} \quad ; \quad Q_j = \frac{\partial S}{\partial P_j} \quad ; \quad K = H + \frac{\partial S}{\partial t} = 0$$

Hamilton's equations give

$$\left\{ \begin{array}{l} \dot{P}_j = -\frac{\partial K}{\partial Q_j} = 0 \\ \dot{Q}_j = \frac{\partial K}{\partial P_j} = 0 \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} P_j = \alpha_j = \text{const.} \\ Q_j = \beta_j = \text{const.} \end{array} \right.$$

The problem has the solution

$$\left\{ \begin{array}{lcl} q_i(t) & = & q_i(\underset{\sim}{\beta}, \underset{\sim}{\alpha}, t) \\ p_i(t) & = & p_i(\underset{\sim}{\beta}, \underset{\sim}{\alpha}, t) \end{array} \right. \qquad \underset{\sim}{\alpha} \text{ and } \underset{\sim}{\beta} \text{ from the initial conditions}$$

Method 2 - transformation directly through $W(\underset{\sim}{q}, \underset{\sim}{\alpha})$

$$p_i = \frac{\partial W}{\partial q_i} \quad ; \quad Q_j = \frac{\partial W}{\partial P_j} \quad ; \quad K = H = E(\underset{\sim}{P}) = E(\underset{\sim}{\alpha})$$

Hamilton's equations give

$$\left\{ \begin{array}{l} \dot{P}_j = -\frac{\partial K}{\partial Q_j} = 0 \\ \dot{Q}_j = \frac{\partial K}{\partial P_j} = \frac{\partial E}{\partial \alpha_j} = v_j = \text{const.} \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} P_j = \alpha_j = \text{const.} \\ Q_j = v_j t + \beta_j \end{array} \right.$$

The problem has the solution

$$\left\{ \begin{array}{lcl} q_i(t) & = & q_i(\underset{\sim}{v} t + \underset{\sim}{\beta}, \underset{\sim}{\alpha}) \\ p_i(t) & = & p_i(\underset{\sim}{v} t + \underset{\sim}{\beta}, \underset{\sim}{\alpha}) \end{array} \right. \qquad \underset{\sim}{\alpha} \text{ and } \underset{\sim}{\beta} \text{ from the initial conditions}$$

Action angle variables

1. Choose

$$P = J \equiv \oint p dq \qquad ; \qquad Q = w$$

2. Use Hamilton-Jacobi's *characteristic* (time independent) equation to get $W(q, \alpha)$.

3. Replace p with $\frac{\partial S}{\partial q}$ in the expression for J and integrate

4. Solve for E from this equation and we have our new Hamilton function $K = E(J)$

5. Hamilton's equations then give

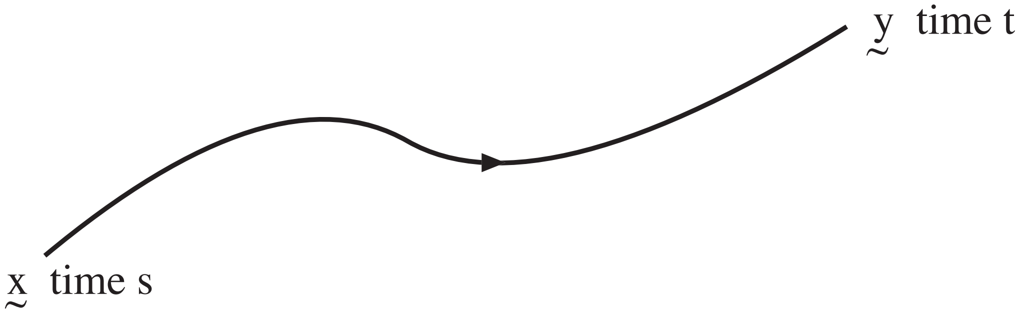
$$\begin{cases} \dot{J} = -\frac{\partial K}{\partial w} = 0 \\ \dot{w} = \frac{\partial K}{\partial J} = \nu = \text{frequency} = \frac{\omega}{2\pi} \end{cases}$$

   and we get get the angular frequency *without either deriving the canonical transformation of motion explicity!*

Remark. Can be generalized to multiple periodic separable systems.

The phase space

Def. The phase space $\mathbf{P}$ to a canonical system is the space of points $\underset{\sim}{x} = \{\underset{\sim}{q}, \underset{\sim}{p}\}$.



The equations of motion can be written

$$\dot{\underset{\sim}{x}} = \underset{\sim}{\mathcal{F}} = \left\{ \frac{\partial H}{\partial \underset{\sim}{p}}, -\frac{\partial H}{\partial \underset{\sim}{q}} \right\}$$

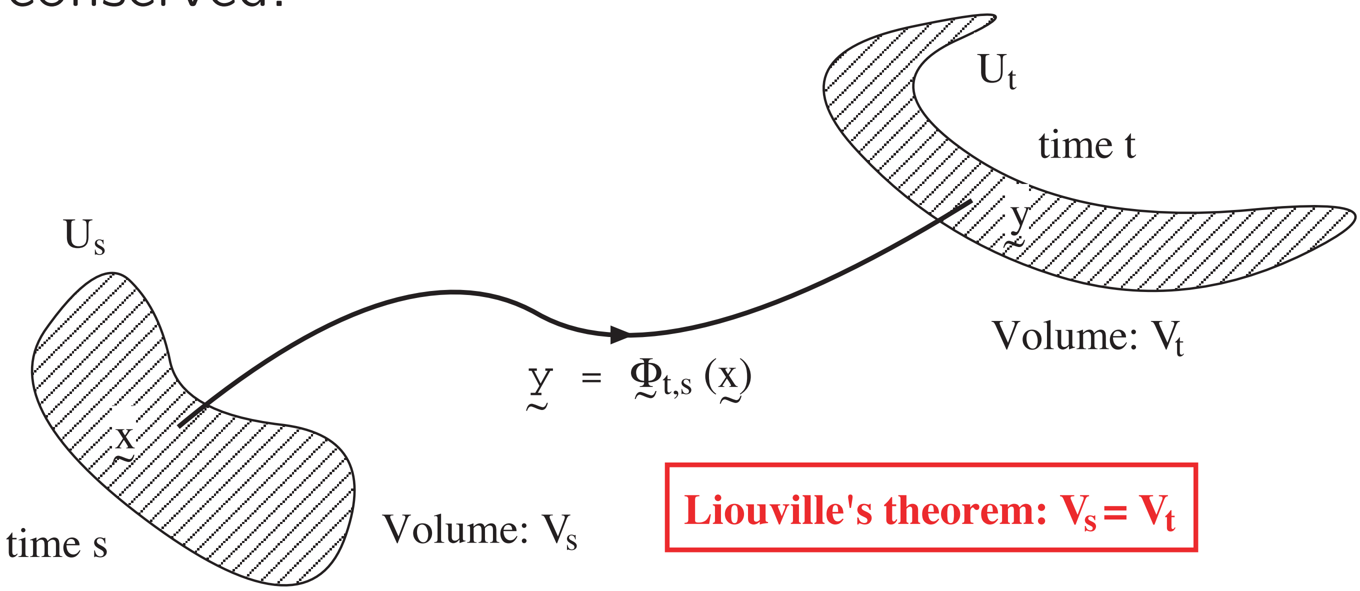
In general, we can write the solution as $\underset{\sim}{\Phi}_{t,s}(\underset{\sim}{x})$

Flows in phase space

For a set of initial conditions $\{\underset{\sim}{x}\}$, the solutions $\underset{\sim}{\Phi}_{t,s}(\underset{\sim}{x})$ describes a *flow* in phase space

Liouville's theorem

Consider a set of initial conditions at time s. This set occupies region U_s with volume V_s. At some later time t, these points have moved to a region U_t with volume V_t. Liouville's theorem them states that $V_t = V_s$, i.e. the volume in phase space is conserved.



The volume in phase space is conserved!

Poisson brackets

Def. If f and g are functions of the canonical variables $\underset{\sim}{q}$ and $\underset{\sim}{p}$, then

$$[f,g] = \sum_i^f \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

is their *Poisson bracket*.

Theorem. The transformation $(\underset{\sim}{q}, \underset{\sim}{p}) \to (\underset{\sim}{Q}, \underset{\sim}{P})$ is canonical if and only if

$$\begin{cases} [Q_i, Q_j] = [P_i, P_j] = 0 \\ [Q_i, P_j] = \delta_{ij} \end{cases}$$

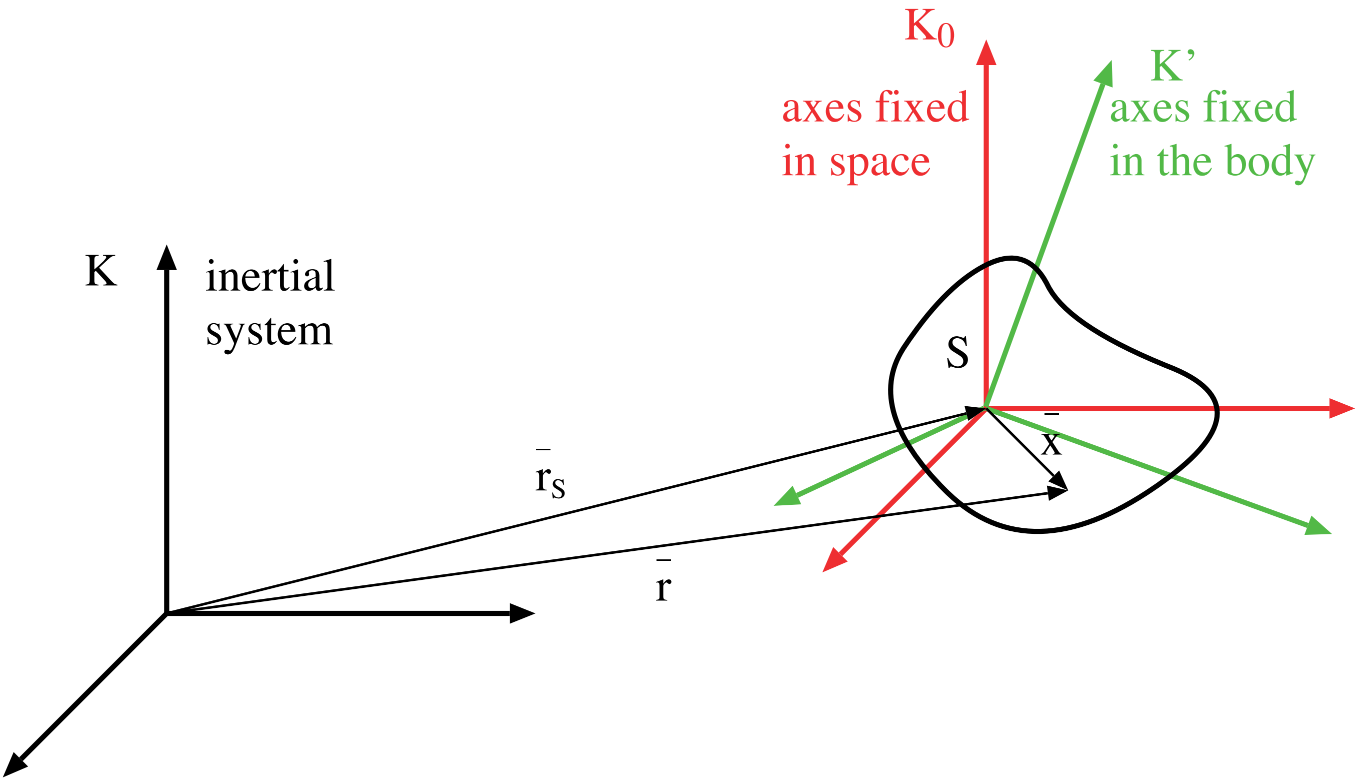
Remark 1. The canonical equations can be written

$$\begin{cases} \dot{q}_k & = & \frac{\partial H}{\partial p_k} & = & [q_k, H] \\ \dot{p}_k & = & -\frac{\partial H}{\partial q_k} & = & [p_k, H] \end{cases}$$

Remark 2 If $g = g(\underset{\sim}{q}, \underset{\sim}{p}, t)$ we have that

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$$

Rigid body motion



Introduce a reference point S in the body (often chose to be the center of mass) and divide position vectors into two parts,

$$\vec{r} = \vec{r}_S + \vec{x}$$

The velocity is given by

$$\vec{v} = \vec{V} + \vec{\omega} \times \vec{x}$$

with $\vec{\omega}$ = the angular frequency.

Kinetic energy and the tensor of inertia

Now let S be the center of mass!

The kinetic energy can then be written

$$T = \frac{1}{2} M V^2 + T_{\text{rot}}$$

with

$$T_{\text{rot}} = \frac{1}{2} \vec{\omega} \cdot \vec{\vec{I}} \cdot \vec{\omega}$$

where $\vec{\vec{I}}$ is *the tensor of inertia*,

$$\vec{\vec{I}} = \begin{cases} \sum_i m_i \left[\vec{x}_i \cdot \vec{x}_i - \vec{x}_i \vec{x}_i \right] & , \ \text{discrete case} \\ \int \left[\vec{x} \cdot \vec{x} - \vec{x}\vec{x} \right] \rho(\vec{x}) d^3 x & , \ \text{continuous case} \end{cases}$$

Remark.

$\vec{x}_i \cdot \vec{x}_i$ - usual scalar product

$\vec{x}_i \vec{x}_i$ - dyadic product

In the base system $(\hat{x}, \hat{y}, \hat{z})$, $\vec{\vec{I}}$ can be represented by a matrix,

$$\vec{\vec{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Properties of the inertia tensor

- $\vec{\vec{I}}$ is linear,

$$\vec{\vec{I}} = \vec{\vec{I}}_1 + \vec{\vec{I}}_2$$

- $\vec{\vec{I}}$ is symmetric,

$$I_{kl} = I_{lk}$$

- It is always possible to rotate K' such that $\vec{\vec{I}}$ is diagonal,

$$\vec{\vec{I}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

  The axes in the new system are called *principal axes*. There are some symmetry rules that can be used to find a principal system easily.

- If $\vec{\vec{I}}$ is calculated in a system fixed in the body, then $\vec{\vec{I}}$ is constant.

Angular momentum

The angular momentum is given by

$$\vec{L} = \underbrace{\vec{r}_S \times M\dot{\vec{r}}_S}_{\text{angular momentum for CM}} + \underbrace{\vec{L}_{\text{rel}}}_{\text{relative angular momentum}}$$

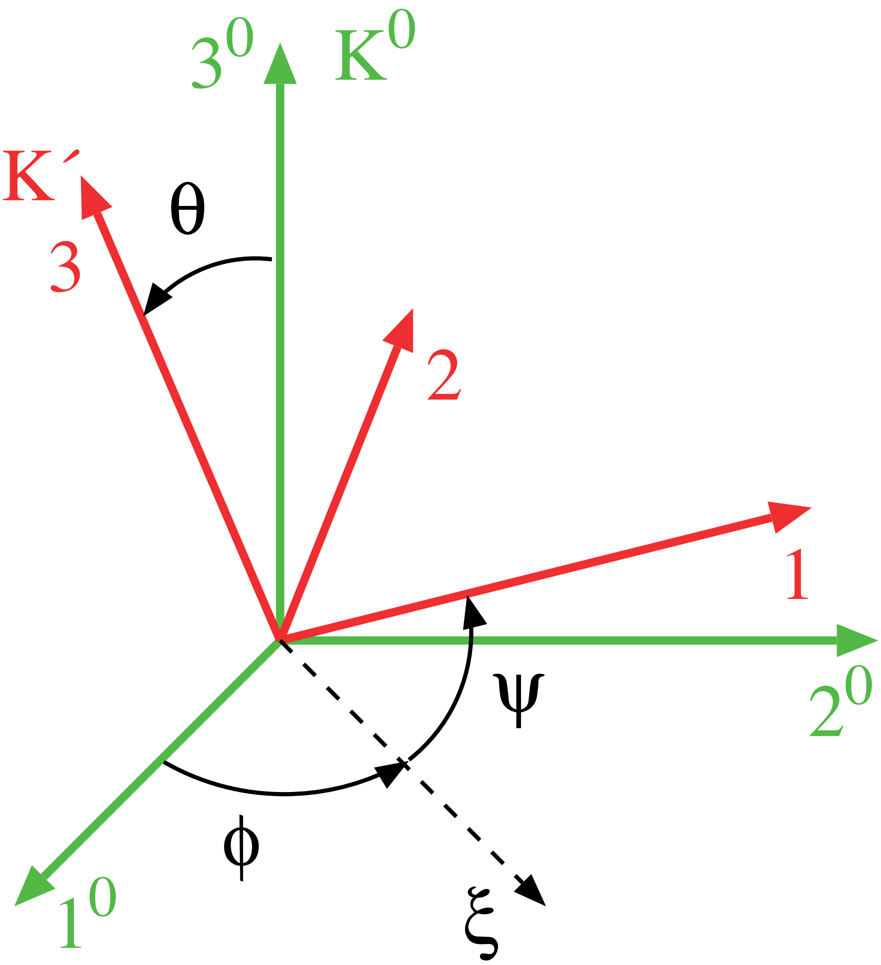
where

$$\vec{L}_{\text{rel}} = \vec{\vec{I}} \cdot \vec{\omega}$$

$$\Rightarrow \quad T_{\text{rot}} = \frac{1}{2}\vec{\omega} \cdot \vec{\vec{I}} \cdot \vec{\omega} = \frac{1}{2}\vec{\omega} \cdot \vec{L}_{\text{rel}}$$

Remark. $\vec{L}_{\text{rel}}$ is dynamically most interesting as it does not depend on our choice of K.

Euler angles



$$\mathbf{R}(t) = \mathbf{R}_3(\psi)\mathbf{R}_\xi(\theta)\mathbf{R}_{3^0}(\phi)$$

$$\begin{cases} \mathbf{R}_{3^0}(\phi) & - \quad \text{rotate around the } 3^0\text{-axis an angle } \phi \\ \mathbf{R}_\xi(\theta) & - \quad \text{rotate around the } \xi\text{-axis and angle } \theta \\ \mathbf{R}_3(\psi) & - \quad \text{rotate around the 3-axis and angle } \psi \end{cases}$$

Remark. In quantum mechanics, one usually makes another choice of rotations and angles, (α, β, γ).

The equations of motion

For the center of mass, we have

$$\frac{d}{dt}\vec{P} = \sum_i \vec{F}_i$$

For the rotation *Euler's dynamical equations hold*

$$\boxed{\vec{N}_0 = \vec{\vec{I}}' \cdot \dot{\vec{\omega}}' + \vec{\omega}' \times \vec{\vec{I}}' \cdot \vec{\omega}'}$$

where

$$\vec{N}_0 = \sum_i \vec{r}_i \times \vec{F}_i$$

is the sum of the external torques. A prime indicates that the quantity is calculated in the system K' with axes fixed in the body.

In a principal system, we get

$$\begin{cases} N_x &= I'_{xx}\dot{\omega}'_x + (I'_{zz} - I'_{yy})\omega'_y\omega'_z \\ N_y &= I'_{yy}\dot{\omega}'_y + (I'_{xx} - I'_{zz})\omega'_z\omega'_x \\ N_z &= I'_{zz}\dot{\omega}'_z + (I'_{yy} - I'_{xx})\omega'_x\omega'_y \end{cases}$$

$\vec{\omega}$ can be expressed in terms of the Euler angles,

$$\begin{cases} \omega'_x &= \dot{\theta}\cos\psi &+ \dot{\phi}\sin\theta\sin\phi \\ \omega'_y &= -\dot{\theta}\sin\psi &+ \dot{\phi}\sin\theta\cos\phi \\ \omega'_z &= \dot{\phi}\cos\theta &+ \dot{\psi} \end{cases}$$

Connections to quantum mechanics I

The correspondance principle

$$\begin{aligned} p_i = \frac{\partial L}{\partial \dot{q}_i} &\rightarrow -i\hbar \frac{\partial}{\partial q_i} \\ [u, v] &\rightarrow \frac{1}{i\hbar}[\hat{u}, \hat{v}] = \frac{1}{i\hbar}(\hat{u}\hat{v} - \hat{v}\hat{u}) \end{aligned}$$

For a canonical variable, we have

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$$

In quantum mechanics, this becomes *Heissenberg's equations of motion*

$$\frac{d\hat{g}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{g}\right] + \frac{\partial \hat{g}}{\partial t}$$

which describes how an operator evolves in time in the so-called *Heissenberg picture*.

Connections to quantum mechanics II

Start from *the Schrödinger equation*

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \quad , \quad \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U$$

Make the Ansatz

$$\psi(q,t) = Ae^{\frac{i}{\hbar}S(q,t)}$$

$$\Downarrow$$

$$\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + U\right] + \frac{\partial S}{\partial t} = \frac{i\hbar}{2m}\left(\frac{\partial^2 S}{\partial q^2}\right)$$

The right-hand side can be neglected if

$$\hbar\left(\frac{\partial^2 S}{\partial q^2}\right) \ll \left(\frac{\partial S}{\partial q}\right)^2$$

which can be rewritten as

$$\frac{\partial p/\partial q}{p/(\lambda/2\pi)} \ll 1.$$

I.e., if the wavelength is so short that the momentum is almost constant over one wavelength, then we get back Hamilon-Jacobi's equation from the Schrödinger equation.