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Exam in Analytical Mechanics, 5p

August 25, 2006

9–15

5 problems on 6 hours. Each problem gives a maximum of 5 points.
Write your name on each sheet of paper!
If you want your result by e-mail, write your e-mail address on the first page.
Allowed aids of assistance: Physics Handbook and attached collection of formulae.

1. Consider a rigid body.

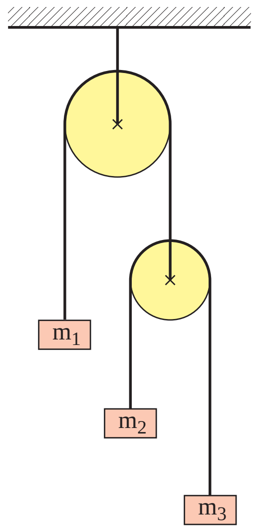
a) Show that the angular frequency ω is unique, i.e. that it is independent of which reference point we choose in a rotating rigid body. (3p)

b) Show that for the components of the tensor of inertia, $I_{zz} \leq I_{xx} + I_{yy}$. For which bodies do we get equality? (2p)

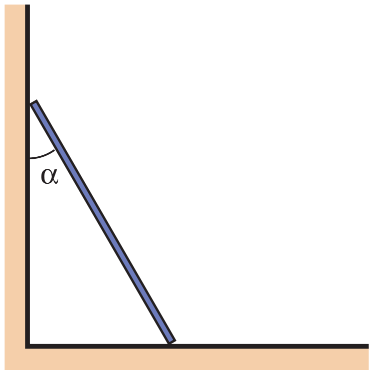
If you have passed on the hand-in exercises you do not need to do problem 2 below, you will get full points on it anyway.

2. A double Atwood machine (see figure) consists of thee masses m_1, m_2 and m_3 connected with massless threads. The threads run over two massless pulleys which can rotate without friction around the symmetry axes. The masses are affected by the gravitational force downwards in the figure.

Write down the equations of motion for the system and solve these. The motion of the three masses can be assumed to take place in the vertical direction only. (5p)



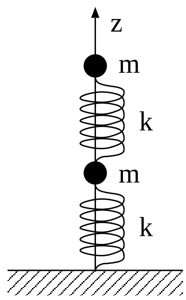
3. A ladder stands on a terrace leaning with an angle α (see figure) towards a newly oiled wall (against which the friction is negligible). It suddenly starts to rain, whereby the friction between the ladder and the terrace disappears. The ladder then starts sliding down towards the terrace under the influence of gravity. The ladder has the length l, mass m and can be approximated with a thin homogeneous rectangular plate.



a) Derive the equations of motion for the ladder as long as it is in contact with the wall. (3p)

b) Will the ladder during its fall lose contact with the wall? If that is the case, at which angle does this happen? (2p)

4. Consider a system with two masses m and two springs with the spring constants k and the natural length a (see figure). The masses can move vertically along the z-axis and are thus affected by both the forces from the springs and the gravitational force. Determine the angular frequencies of the system.



Hint: The solutions to a system of second order differential equations on the form

$$\underset{\sim}{\ddot{y}} = \mathbf{A}\underset{\sim}{y} + \underset{\sim}{B}$$

can be written as $\underset{\sim}{y} = \underset{\sim}{y}_h + \underset{\sim}{y}_p$ *where* $\underset{\sim}{y}_p$ *is the particular solution to the equation above and* $\underset{\sim}{y}_h$ *is the solution to the homogeneous equation* $\underset{\sim}{\ddot{y}} = \mathbf{A}\underset{\sim}{y}$. $\underset{\sim}{y}_h$ *is given by a linear combination of the solutions that are obtained by putting the Ansatz*

$$\underset{\sim}{y} = \underset{\sim}{a}\cos(\omega t + \delta)$$

into the homogeneous equation.

5. Consider a particle with mass m in a homogeneous gravitational field. The particle can move both horizontally (in the x direction) and vertically (in the y direction).

a) Write down and solve Hamilton-Jacobi's equation for this system, i.e. derive the generating function (action function) $S(x, y, \underset{\sim}{\alpha}, t)$ for this system. (3p)

b) Use the generating function S you derived in a) to transform to new canonical variables $(Q_1, Q_2, P_1 = \alpha_1, P_2 = \alpha_2)$. Solve Hamilton's equations for these variables and transform back to the original variables to find the solution $(x(t), y(t))$ if the particle at $t = 0$ starts on height h with a purely horizontal velocity v_0, i.e. $x(0) = 0$, $\dot{x}(0) = v_0$, $y(0) = h$, $\dot{y}(0) = 0$. (2p)

Hint: There are many possible solutions S to Hamilton-Jacobi's equation. One possible solution for this problem is

$$S(x, y, \alpha_1, \alpha_2, t) = \alpha_2 x + \frac{1}{3m^2 g}\left(2m\alpha_1 - \alpha_2^2 - 2m^2 g y\right)^{\frac{3}{2}} - \alpha_1 t$$

where α_1 annd α_2 are the new constant canonical momenta. If you don't succeed in deriving a generating function in a), you can use this one to solve part b) of the problem.

Good luck!

The solutions will be posted after the exam. They will also be available on `http://www.physto.se/~edsjo/teaching/am/index.html`.