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Exam in Analytical Mechanics, 5p

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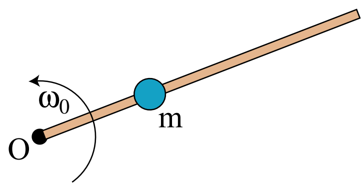
9–15

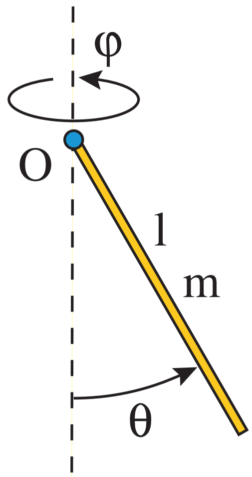
5 problems on 6 hours. Each problem gives a maximum of 5 points.
Write your name on each sheet of paper!
If you want your result by e-mail, write your e-mail address on the first page.
Allowed aids of assistance: Physics Handbook and attached collection of formulae.

1. A particle with mass m can move without friction along a straight rod. The rod rotates in the horizontal plane around the fixed point O with a constant angular velocity ω_0. Determine the mass m's motion if it at $t = 0$ starts a distance a from O and without velocity along the rod. Only the motion as long as the mass m has contact with the rod needs to be considered. (5p)



If you have passed on the hand-in exercises you do not need to do problem 2 below, you will get full points on it anyway.

2. A straight, homogeneous thin rod with mass m and length l can rotate without friction around a fixed joint at O (see figure).



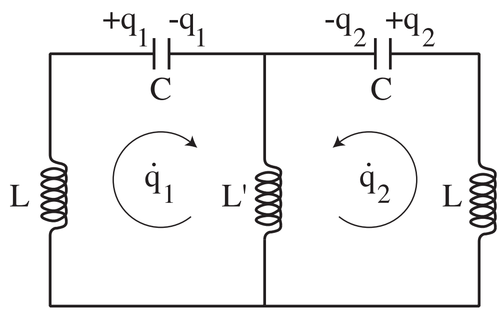
a) If θ is the angle from the vertical axis and φ is the azimuthal angle for the rotation around the same axis (see figure), show that the kinetic energy is given by

$$T = \frac{ml^2}{6}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right)$$

(2p)

b) Initially, the rod is moving horizontally (i.e. with $\theta = \pi/2$ and $\dot{\theta} = 0$) with the angular velocity $\dot{\varphi} = \omega_0$ around the vertical axis. Due to the gravitational force, the rod will start turning downwards in the subsequent motion. Determine $\dot{\varphi}$ as a function of θ and determine the turn-around point for the θ-motion. (3p)

3. The Lagrange formalism can be used on other problems than mechanical ones. E.g. it can be used on electrical circuits. To a coil with the inductance L, we can associate a kinetic energy $\frac{1}{2}L\dot{q}^2$ where $\dot{q}$ is the charge flow through the coil. To a capacitor, we can in a similar manner associate a potential energy $\frac{1}{2}\frac{q^2}{C}$ where q is the charge (per plate) and C is the capacitance. Consider the circuit in the figure (the arrows indicate the direction of the charge flow).



a) Write down the Lagrangian for this circuit and derive the equations of motion for the charges q_1 and q_2. (2p)

b) Determine the eigenfrequencies of the circuit. (3p)
Hint: A suitable Ansatz could be

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} \quad , \quad a_1,\ a_2 \text{ and } \omega = \text{constants}$$

4 a) Consider the functional

$$I[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x)dx$$

where $y' = dy/dx$ and f is a function of y, y' and x. x_1 and x_2 are two arbitrary (but fixed) endpoints with $y(x_1) = y_1$ and $y(x_2) = y_2$. Show that if $I[y]$ assumes a stationary value, then y satisfies the Euler equation for the variational problem,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

(3p)

b) Consider two points (x_0, y_0) and (x_1, y_1) in the xy-plane. Show that the shortest path between these two points is a straight line. (2p)
Hint: The line element is given by $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$.

5. Consider a particle with mass m that can move in one dimension and that is described by the Hamiltonian (k is a constant)

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

a) Write down or derive a canonical non-trivial transformation (i.e. not the identity transformation or similar) of your own choice for this system. Show that the transformation is canonical (if it is not obvious from the way you have derived the transformation). (3p)

b) Perform the transformation in a) and solve the equations of motion for this transformed system. Then transform back to our original canonical variables $\{q, p\}$, and write down how the solution $\{q(t), p(t)\}$ looks like. (2p)

Good luck!

The solutions will be posted after the exam. They will also be available on `http://www.physto.se/~edsjo/teaching/am/index.html`.