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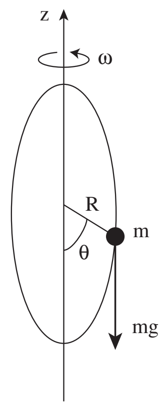
Exam in Analytical Mechanics, 5p

August 20, 2004

9–15

5 problems on 6 hours. Each problem gives a maximum of 5 points.
Write your name on each sheet of paper!
If you want your result by e-mail, write your e-mail address on the first page.
Allowed aids of assistance: Physics Handbook and attached collection of formulae.

1. A mass m can move without friction along a circular wire (see figure). The wire rotates around the vertical diameter (the z axis) with a constant angular velocity ω. The mass m is affected by the gravitational force downwards in the figure. Let θ be the angle between the vertical direction and the mass m according to the figure.



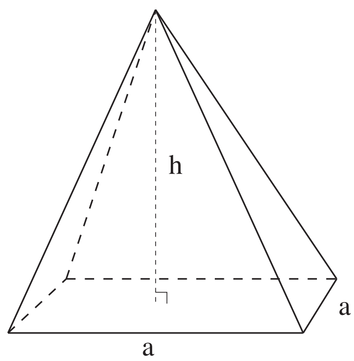
a) Derive the equation of motion for θ. (2p)

b) For low angular velocities, $\theta = 0$ is a stable equilibrium point, whereas it is unstable for high angular velocities. Determine the critical angular velocity ω_c that separates these two cases. (2p)

c) When $\omega < \omega_c$, only $\theta = 0$ and $\theta = \pi$ are equilibrium points, whereas when $\omega > \omega_c$ there is one more equilibrium point. Determine this point! (1p)

If you have passed on the hand-in exercises, you don't have to do problem 2 below, as you will get full points for it anyway.

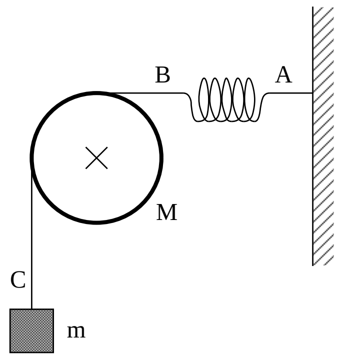
2. Consider a pyramid with mass m and whose base is quadratical with side length a and whose height is h (se figure).



a) Show that the center of mass of the pyramid is at height $h/4$ from its base. (2p)

b) Introduce a suitable coordinate system and derive the tensor of inertia with respect to the pyramid's center of mass. (3p)

3. A cylindrical shell with radius R and mass M can rotate without friction around its symmetry axis. The symmetry axis is horizontal and parallel with a vertical wall. A spring AB with the spring constant k is fastened to the wall and to a thin, flexible, inelastic thread BC that goes over the cylinder, perpendicular to the symmetry axis. No sliding occurs between the thread and the cylinder. In point C on the thread, a mass m is hanging (affected by gravitation). The thread and the spring has negligible masses and the mass m can be assumed to move only vertically.



a) Determine the equilibrium point. (2p)

b) Determine the motion of the system if it is released from rest in a state where the spring assumes its natural length. (3p)

4. a) Define the concept *canonical transformation* and describe how a generating function can be used to generate the transformation. (2p)

b) Start from Hamilton's variational principle, $\delta \int [\sum_i p_i \dot{q}_i - H(\underset{\sim}{q}, \underset{\sim}{p}, t)]dt = 0$, and show that a generating function $S(\underset{\sim}{q}, \underset{\sim}{P}, t)$ can generate a canonical transformation. Also derive the relations sthat then hold between the old variables $\{\underset{\sim}{q}, \underset{\sim}{p}\}$ and the new variables $\{\underset{\sim}{Q}, \underset{\sim}{P}\}$. (3p)

Hint: Note that $\frac{d}{dt} \sum_i Q_i P_i$ *can be subtracted or added to the Hamilton function without changing the equations of motion.*

5. a) Consider an autonomous (time independent) system that is described by a Lagrangian $L(\underset{\sim}{q}, \dot{\underset{\sim}{q}})$ which is invariant under some transformation. Write down and prove Noether's theorem for this system. (3p)

b) A particle in three dimensions is described by the Lagrangian

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 + \frac{A}{r + r^3} \qquad ; \qquad A = \text{const.}$$

Show that the angular momentum $\mathbf{L}$ is a constant of motion. (2p)

Good luck!

The solutions will be posted after the exam. They will also be available on `http://www.physto.se/~edsjo/teaching/am/index.html`.