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Exam in Analytical Mechanics, 5p

June 1, 2004

9–15

5 problems on 6 hours. Each problem gives a maximum of 5 points.
Write your name on each sheet of paper!
If you want your result by e-mail, write your e-mail address on the first page.
Allowed aids of assistance: Physics Handbook and attached collection of formulae.

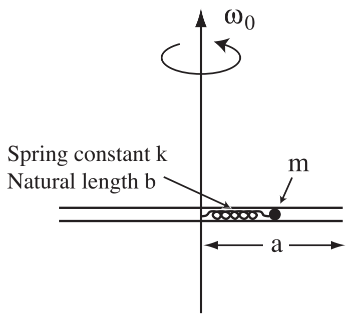
1. a) Define the tensor of inertia for a rigid body with the mass distribution $\rho(\vec{x})$. Show specifically how the different components look like in a cartesian coordinate system. (1p)

b) Show that if the body is mirror symmetric in the xy-plane, then $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$. (2p)

c) Show that for the components of the tensor of inertia, we have that $I_{zz} \leq I_{xx} + I_{yy}$. For which bodies does the equality hold? (2p)

If you have passed on the hand-in exercises, you don't have to do problem 2 below, as you will get full points for it anyway.

2. A mass m can move without friction in a cylindrical tube with length $2a$. The tube rotates with the angular frequency ω_0 around a rotational axis perpendicular to the tube and that goes through the tube's center of mass (see figure). The mass m is attached to the rotational axis via a mass less spring with the natural length b and spring constant k.



a) Derive the equations of motion of the mass m as long as the mass is inside the tube. (2p)

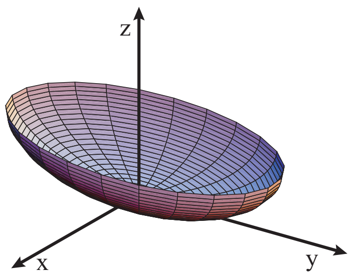
b) How does the motion look like (while the mass m is inside the tube)? Sketch the different types of motion we can get and determine a condition on the spring constant k and the angular frequency ω_0 that distinguish the different types of motion. (2p)

c) Assume that the spring constant is given by $k = 2m\omega_0^2$. If the mass m at $t = 0$ is at the distance b from the rotational axis and without motion along the tube, derive the full solution to the equations of motion. (1p)

3. A bathtub has the shape of a half ellipsoid, where the height, z, is given by

$$z = c - c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

where a, b and c are constants. You have just taken a bath and flushed the water when you drop the soap in the tub. The soap will then describe small oscillations around the equilibrium point at the bottom of the tub. Determine the angular frequencies of these oscillations! The friction between the soap and the tub can be neglected.



4. Start from Hamilton's principle (or, if you prefer, from the principle of virtual work or d'Alembert's principle) and derive Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad ; \quad \forall\, k = 1, \ldots, f \quad ; \quad f = \text{number of degrees of freedom}$$

(5p)

5. By using a canonical transformation of class B, we can derive the Hamilton-Jacobi equation for the generating function $S(\underset{\sim}{q}, \underset{\sim}{P}, t)$ such that the new Hamiltonian is identically equal to zero.

a) Show that one, in the same way, can use a transformation of class C with a generating function $U(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ such that the new Hamiltonian is identically equal to zero. Which differential equation does U have to fulfill? (This equation is called the Hamilton-Jacobi equation in the momentum representation.) (2p)

b) Use the equation you derived in a) to find the generating function $U(Q, p, t)$ for a particle that can move vertically in a homogenous gravitational field, i.e. with the Hamiltonian

$$H = \frac{p^2}{2m} + mgq$$

where q is the height above the horizontal plane. Then use this U to generate a canonical transformation which makes the problem trivial to solve. Solve the equations of motion for the new canonical variables and then determine the motion $\{q(t), p(t)\}$ if the initial conditions are that $p(t = 0) = mv_0$ and $q(t = 0) = 0$.

Comment: If you prefer (or have not managed to solve part a), you can instead use the usual Hamilton-Jacobi equation for $S(q, P, t)$ to solve part b. (3p)

Good luck!

The solutions will be posted after the exam. They will also be available on `http://www.physto.se/~edsjo/teaching/am/index.html`.