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Exam in Analytical Mechanics, 5p

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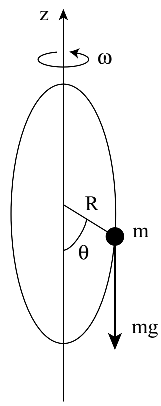
9–15

5 problems on 6 hours. Each problem gives a maximum of 5 points.
Write your name on each sheet of paper!
If you want your result by e-mail, write your e-mail address on the first page.
Allowed books: Physics Handbook.

1. A mass m can move without friction along a circular wire (see figure). The wire rotates around the vertical diameter (the z axis) with a constant angular velocity ω. The mass m is affected by the gravitational force downwards in the figure. Let θ be the angle between the vertical direction and the mass m according to the figure.



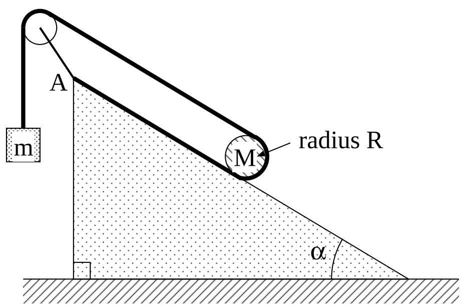
a) Derive the equation of motion for θ. (2p)

b) For low angular velocities, $\theta = 0$ is a stable equilibrium point, whereas it is unstable for high angular velocities. Determine the critical angular velocity ω_c that separates these two cases. (2p)

c) When $\omega < \omega_c$, only $\theta = 0$ and $\theta = \pi$ are equilibrium points, whereas when $\omega > \omega_c$ there is one more equilibrium point. Determine this point! (1p)

If you have passed on the hand-in exercises, you don't have to do problem 2 below. You will get full points for it anyway.

2. A homogenous solid cylinder with mass M and radius R can roll without friction on a fixed wedge with angle α (see figure). Around the cylinder, a thin thread (with negligible mass) is wrapped. One end of the thread is attached to the point A, whereas the other end is attached to a mass m. The thread runs over a frictionless and massless wheel at A. The masses m and M are affected by gravity downwards in the figure.



a) Derive and solve the equation of motion for the mass m. (3p)

b) Determine the angle α for which the system is in equilibrium. (2p)

3. a) If f, g and h are functions of the canonical variables, show the following properties for the Poisson brackets,

$$\begin{aligned}\{f, gh\} &= g\{f, h\} + \{f, g\}h \\ \{fg, h\} &= f\{g, h\} + \{f, h\}g\end{aligned}$$

(2p)

b) Consider a particle in three dimensions that move in the potential

$$U = \alpha z^2 e^{\beta x^2 + \gamma y^2} \quad ; \quad \alpha, \beta, \gamma = \text{constants}, \quad \alpha \neq 0.$$

Determine a condition on β and γ such that the z component of the angular momentum is conserved (for arbitrary initial conditions). (3p)

4. a) Consider the functional

$$I[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx$$

where $y' = dy/dx$ and f is a function of y, y' och x. x_1 and x_2 are two arbitrary (but fixed) end points with $y(x_1) = y_1$ and $y(x_2) = y_2$. Show that if $I[y]$ assumes an extremum, then y satisfies Euler's equation for the variational problem,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$ (3p)

b) Consider two points (x_0, y_0) and (x_1, y_1) in the xy plane. Show that the shortest path between these two points is a straight line. (2p)
Hint: The line element is given by $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$.

5. By using a canonical transformation of class B, we can derive the Hamilton-Jacobi equation for the generating function $S(\underset{\sim}{q}, \underset{\sim}{P}, t)$ such that the new Hamiltonian is identically equal to zero.

a) Show that one, in the same way, can use a transformation of class C with a generating function $U(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ such that the new Hamiltonian is identically equal to zero. Which differential equation does U have to fulfill? (This equation is called the Hamilton-Jacobi equation in the momentum representation.) (2p)

b) Use the equation you derived in a) to find the generating function $U(Q, p, t)$ for a particle that can move vertically in a homogenous gravitational field, i.e. with the Hamiltonian

$$H = \frac{p^2}{2m} + mgq$$

where q is the height above the horizontal plane. Then use this U to generate a canonical transformation which makes the problem trivial to solve. Solve the equations of motion for the new canonical variables and then determine the motion $\{q(t), p(t)\}$ if the initial conditions are that $p(t = 0) = mv_0$ and $q(t = 0) = 0$. (3p)

Good luck!

Solutions will eventually be posted as well as be available on `http://www.physto.se/~edsjo/teaching/am/index.html`.

Collection of formulae

Canonical transformations

Class A. $\Phi = \Phi(\underset{\sim}{q}, \underset{\sim}{Q}, t)$ - generating function

$$p_i = \frac{\partial \Phi}{\partial q_i} \quad ; \quad P_j = -\frac{\partial \Phi}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial \Phi}{\partial t}$$

Class B. $S = S(\underset{\sim}{q}, \underset{\sim}{P}, t)$ - generating function

$$p_i = \frac{\partial S}{\partial q_i} \quad ; \quad Q_j = \frac{\partial S}{\partial P_j} \quad ; \quad \tilde{H} = H + \frac{\partial S}{\partial t}$$

Class C. $U = U(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial U}{\partial p_i} \quad ; \quad P_j = -\frac{\partial U}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial U}{\partial t}$$

Class D. $V = V(\underset{\sim}{P}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial V}{\partial p_i} \quad ; \quad Q_j = \frac{\partial V}{\partial P_j} \quad ; \quad \tilde{H} = H + \frac{\partial V}{\partial t}$$

Noether's theorem

If the Lagrangian $L(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ describes an autonomous system which is invariant under the transformation $\underset{\sim}{q} \to h^s(\underset{\sim}{q})$ where s is a real continuous parameter such that $h^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ is the identity transformation, then

$$I(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h^s(q_i) \Big|_{s=0}$$

is a constant of motion.