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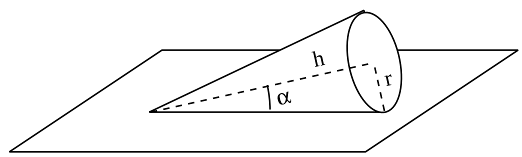
Exam in Analytical Mechanics, 5p

May 27 2000

9–15

5 problems in 6 hours. Each problem can give 5 points.
Write your name on every sheet!
If you want your results by e-mail, write your e-mail address on the first page.
Allowed books: Physics Handbook

1. A homogenous cone with mass m, height h and top angle 2α is rolling without slipping on a plane. The angular velocity is ω_0. Calculate the kinetic energy. (5p)



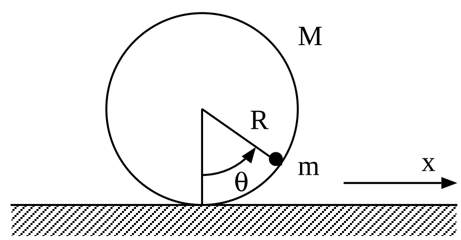
2. Consider a particle in three dimensions with mass m which is moving in the potential

$$U(\mathbf{r}) = \alpha z^2 e^{\beta(x^2+y^2)} \quad ; \quad \alpha, \beta = \text{constants}$$

a) Show that the total energy (kinetic + potential) is conserved. (2p)

b) Show that the z-component of the angular momentum is conserved. (3p)

3. A particle with mass m can move on the inside of a cylindrical shell with mass M and radius R. The cylinder can roll without friction on a plane surface and the friction between the mass m and the cylindrical shell can be neglected.



a) Write down the Lagrangian and use it to obtain the equations of motion. The motion can be considered to only take place in the plane of the figure (i.e. the mass m is not moving along the symmetry axis of the cylinder). (3p)

b) If x is the location of the cylinder, solve the equations of motion for $x = x(\theta)$ and write down the solution $x(\theta)$ when the system starts with $x = \dot{x} = \dot{\theta} = 0$ and $\theta = \pi/2$. (2p)

4. a) Write down Liouville's theorem for flows in phase space. (2p)

b) Derive the theorem. (3p)

5. Start from the Schrödinger equation for a particle with mass m in one dimension,

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

with

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{q}) \quad ; \quad \hat{p} = -i\hbar\frac{\partial}{\partial q}$$

a) Let $\Psi(q,t) = Ae^{\frac{i}{\hbar}S^*(q,t)}$ (with A = constant) and derive an equation that resembles the Hamilton-Jacobi equation as much as possible. How can $S^*(q,t)$ be interpreted? (3p)

b) In what limit will the equation derived in a) be identical to the Hamilton-Jacobi equation? Try to discuss/interpret your answer. (2p)

Good luck!

Solutions will be posted after the exam. They will also be available on *http://www.physto.se/~edsjo/teaching/analmek/index.html.*

Collection of formulae

Canonical transformations

Typ A. $\Phi = \Phi(\underset{\sim}{q}, \underset{\sim}{Q}, t)$ - generating function

$$p_i = \frac{\partial\Phi}{\partial q_i} \quad ; \quad P_j = -\frac{\partial\Phi}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial\Phi}{\partial t}$$

Typ B. $S = S(\underset{\sim}{q}, \underset{\sim}{P}, t)$ - generating function

$$p_i = \frac{\partial S}{\partial q_i} \quad ; \quad Q_j = \frac{\partial S}{\partial P_j} \quad ; \quad \tilde{H} = H + \frac{\partial S}{\partial t}$$

Typ C. $U = U(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial U}{\partial p_i} \quad ; \quad P_j = -\frac{\partial U}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial U}{\partial t}$$

Typ D. $V = V(\underset{\sim}{P}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial V}{\partial p_i} \quad ; \quad Q_j = \frac{\partial V}{\partial P_j} \quad ; \quad \tilde{H} = H + \frac{\partial V}{\partial t}$$

Noether's theorem

If the Lagrangian $L(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ describes an autonomous system which is invariant under the transformation $\underset{\sim}{q} \to h^s(\underset{\sim}{q})$ where s is a real continuous parameter such that $h^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ is the identity transformation, then

$$I(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{ds}h^s(q_i)\bigg|_{s=0}$$

is a constant of motion.