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Solutions to

Exam in Analytical Mechanics, 5p

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Solutions are also available on
http://www.physto.se/~edsjo/teaching/am/index.html.

Problem 1

The problem has one degree of freedom and we can e.g. choose the distance r from O as our generalized coordinate. The kinetic energy is then given by

$$T = \frac{1}{2}m\left(r^2\omega_0^2 + \dot{r}^2\right)$$

which gives us the Lagrangian

$$L = T - V = \frac{1}{2}m\left(r^2\omega_0^2 + \dot{r}^2\right)$$

since we don't have any potential energy. The partial derivatives of L are given by

$$\left\{\begin{array}{lcl} \frac{\partial L}{\partial r} & = & mr\omega_0^2 \\ \frac{\partial L}{\partial \dot{r}} & = & m\dot{r} \end{array}\right.$$

Inserted into Lagrange's equations, we then get

$$m\ddot{r} = mr\omega_0^2 \qquad \Rightarrow \qquad \ddot{r} - \omega_0^2 r = 0 \tag{1}$$

This equation we solve with the Ansatz

$$r(t) = Ae^{\omega t} \tag{2}$$

where A and ω are constants. Inserted into Eq. (1) we get

$$A\omega^2 e^{\omega t} - \omega_0^2 A e^{\omega t} = 0 \qquad \Rightarrow \qquad \omega^2 = \omega_0^2 \qquad \Rightarrow \qquad \omega = \pm\omega_0$$

Our general solution is then a superposition of these solutions (and no particular solution as Eq. (1) is homogenous). The general solution is therefore

$$r(t) = A_1 e^{\omega_0 t} + A_2 e^{-\omega_0 t}$$

The initial condition $r(0) = a$ gives

$$A_1 + A_2 = a$$

The initial condition $\dot{r}(0) = 0$ gives

$$A_1\omega_0 - A_2\omega_0 = 0 \qquad \Rightarrow \qquad A_1 = A_2$$

which together give

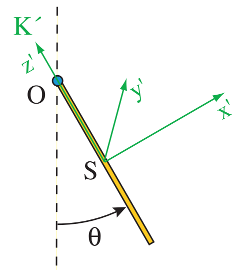
$$A_1 = \frac{a}{2} \qquad ; \qquad A_2 = \frac{a}{2}$$

which finally give us the solution

$$r(t) = \frac{a}{2}\left(e^{\omega_0 t} + e^{-\omega_0 t}\right)$$

Problem 2

a) The kinetic energy consists of translational energy for the center of mass and rotational energy for the rotation around the center of mass,



$$T = \frac{1}{2}m\left[\left(\frac{l}{2}\dot{\theta}\right)^2 + \left(\frac{l}{2}\sin\theta\dot{\varphi}\right)^2\right] + \frac{1}{2}\omega\cdot\mathbf{I}\cdot\omega$$

Let us calculate the last term, the rotational energy in a body-fixed system K' with origo in the center of mass according to the figure. The tensor of inertia for this system is given by

$$\mathbf{I} = \begin{pmatrix} \frac{1}{12}ml^2 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We also need the angular velocity vector ω expressed in the K' system. This one gets contributions from two parts: a) the rotation around the vertical axis and b) from the θ rotation,

$$\omega = \dot{\varphi}\left(\sin\theta\hat{\mathbf{x}}' + \cos\theta\hat{\mathbf{z}}'\right) - \dot{\theta}\hat{\mathbf{y}}'$$

The rotational energy for the rotation around the center of mass is therefore

$$T_{\text{rot}} = \frac{1}{2}\omega\cdot\mathbf{I}\cdot\omega = \frac{1}{2}\frac{1}{12}ml^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) = \frac{1}{24}ml^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right)$$

The total kinetic energy is then

$$T = \frac{1}{8}ml^2\dot{\theta}^2 + \frac{1}{8}ml^2\sin^2\theta\dot{\varphi}^2 + \frac{1}{24}ml^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) = \frac{1}{6}ml^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right)$$

b) To get the motion we can use Lagrange's equations. The Lagrangian is given by

$$L = T - U = \frac{1}{6}ml^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}mgl\cos\theta$$

The partial derivatives of L are

$$\left\{\begin{array}{lcl} \frac{\partial L}{\partial\theta} & = & \frac{1}{3}ml^2\sin\theta\cos\theta\dot{\varphi}^2 - \frac{1}{2}mgl\sin\theta \\ \frac{\partial L}{\partial\dot{\theta}} & = & \frac{1}{3}ml^2\dot{\theta} \end{array}\right. \qquad ; \qquad \left\{\begin{array}{lcl} \frac{\partial L}{\partial\varphi} & = & 0 \\ \frac{\partial L}{\partial\dot{\varphi}} & = & \frac{1}{3}ml^2\sin^2\theta\dot{\varphi} \end{array}\right.$$

Inserted into Lagrange's equations, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$, we get the equations of motion,

$$\begin{cases} \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{3}ml^2\sin\theta\cos\theta\dot{\varphi}^2 + \frac{1}{2}mgl\sin\theta = 0 \\ \frac{d}{dt}\left(\frac{1}{3}ml^2\sin^2\theta\dot{\varphi}\right) = 0 \end{cases} \tag{3}$$

The second of of these we can integrate right away and get

$$\sin^2\theta\dot{\varphi} = A = \text{const.} \qquad \Rightarrow \qquad \dot{\varphi} = \frac{A}{\sin^2\theta}$$

The constant A we can determine from the initial conditions $\dot{\varphi}(0) = \omega_0$ and $\theta(0) = \pi/2$, which give $A = \omega_0$. The relationship between $\dot{\varphi}$ as a function of θ is therefore

$$\dot{\varphi} = \frac{\omega_0}{\sin^2\theta} \tag{4}$$

We are also interested in the turn-around point for the θ motion and can either strat from the conservation of energy or the first of the equations of motion, Eq. (3). Let us start from the first of the equations of motion. With Eq. (4), this equation can be written

$$\frac{1}{3}ml^2\ddot{\theta} - \frac{1}{3}ml^2\omega_0^2\frac{\cos\theta}{\sin^3\theta} + \frac{1}{2}mgl\sin\theta = 0$$

Multiply this equation with $\dot{\theta}$ and integrate with respect to time and we get

$$\frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{6}ml^2\omega_0^2\frac{1}{\sin^2\theta} - \frac{1}{2}mgl\cos\theta = B = \text{konst.}$$

The integration constant B can be determined from the initial conditions $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$ which gives $B = \frac{1}{6}ml^2\omega_0^2$. We are now ready to derive the turn-around point for the θ motion and realize that the turn-around points are given when $\dot{\theta} = 0$, which inserted in the equations above give

$$\frac{1}{6}ml^2\omega_0^2\frac{1}{\sin^2\theta} - \frac{1}{2}mgl\cos\theta = \frac{1}{6}ml^2\omega_0^2$$

This equation can be written as

$$ml\cos\theta\left[l\omega_0^2\cos\theta - 3g + 3g\cos^2\theta\right] = 0$$

This equation has two solutions, either that $\cos\theta = 0$, which gives the upper turn-around point $\theta = \pi/2$, i.e. the same as our initial condition. The other solution is obtained when the expression within square brackets is zero, i.e. when

$$\cos^2\theta + \frac{l\omega_0^2}{3g}\cos\theta - 1 = 0$$

This second order equation in $\cos\theta$ is easy to solve and we get the solution

$$\cos\theta = -\frac{l\omega_0^2}{6g} \overset{+}{(-)} \sqrt{1 + \frac{l^2\omega_0^4}{36g}}$$

where only the solution with a + sign in front of the root expression is physical (since $\cos\theta \in [-1, 1]$).

Problem 3

a) With the charge flows according to the figure, the kinetic and potential energies are give by

$$T = \frac{1}{2}L\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}L'\left(\dot{q}_1 + \dot{q}_2\right)^2 \quad ; \quad U = \frac{1}{2}\frac{q_1^2 + q_2^2}{C}$$

which gives the Lagrangian (here written as $\mathcal{L}$ to avoid confusion with the inductance L)

$$\mathcal{L} = T - U = \frac{1}{2}L\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}L'\left(\dot{q}_1 + \dot{q}_2\right)^2 - \frac{1}{2C}\left(q_1^2 + q_2^2\right)$$

The derivatives of the Lagrangian are

$$\left\{\begin{array}{lcl} \frac{\partial\mathcal{L}}{\partial\dot{q}_1} &=& L\dot{q}_1 + L'(\dot{q}_1 + \dot{q}_2) \\ \frac{\partial\mathcal{L}}{\partial\dot{q}_2} &=& L\dot{q}_2 + L'(\dot{q}_1 + \dot{q}_2) \end{array}\right. \quad ; \quad \left\{\begin{array}{lcl} \frac{\partial\mathcal{L}}{\partial q_1} &=& -\frac{1}{C}q_1 \\ \frac{\partial\mathcal{L}}{\partial q_2} &=& -\frac{1}{C}q_2 \end{array}\right.$$

This inserted into Lagrange's equations give us the equations of motion

$$\left\{\begin{array}{l} L\ddot{q}_1 + L'(\ddot{q}_1 + \ddot{q}_2) + \frac{1}{C}q_1 = 0 \\ L\ddot{q}_2 + L'(\ddot{q}_1 + \ddot{q}_2) + \frac{1}{C}q_2 = 0 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} (L+L')\ddot{q}_1 + L'\ddot{q}_2 + \frac{1}{C}q_1 = 0 \\ L'\ddot{q}_1 + (L+L')\ddot{q}_2 + \frac{1}{C}q_2 = 0 \end{array}\right. \tag{5}$$

b) We assume that the solutions will be oscillating and are then given on the form

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

where a_1 and a_2 are constants and ω is our sought after angular frequency. Inserted into Eq. (5) we get

$$\left\{\begin{array}{l} -a_1(L+L')\omega^2 - L'a_2\omega^2 + \frac{1}{C}a_1 = 0 \\ -a_1L'\omega^2 - (L+L')a_2\omega^2 + \frac{1}{C}a_2 = 0 \end{array}\right.$$

Written in matrix form we get

$$\begin{pmatrix} (L+L')\omega^2 - \frac{1}{C} & L'\omega^2 \\ L'\omega^2 & (L+L')\omega^2 - \frac{1}{C} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

which has a non-trivial solution if the determinant of the coefficient matrix is zero, i.e. if

$$0 = \begin{vmatrix} (L+L')\omega^2 - \frac{1}{C} & L'\omega^2 \\ L'\omega^2 & (L+L')\omega^2 - \frac{1}{C} \end{vmatrix} = \left[(L+L')\omega^2 - \frac{1}{C}\right]^2 - L'^2\omega^4$$

which gives us

$$\omega^2(L+L') - \frac{1}{C} = \pm L'\omega^2 \quad \Rightarrow \quad \omega^2\left[(L+L') \mp L'\right] = \frac{1}{C}$$

which gives us the solutions

$$\left\{\begin{array}{lcl} \omega^2 &=& \frac{1}{CL} \\ \omega^2 &=& \frac{1}{C(2L'+L)} \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{lcl} \omega &=& \pm\sqrt{\frac{1}{CL}} \\ \omega &=& \pm\sqrt{\frac{1}{C(2L'+L)}} \end{array}\right.$$

which are our sought after angular frequencies.

Problem 4

a) See Goldstein, chapter 2.2 or the lecture notes.

b) This is easy to solve with variational calculus. The distance between (x_0, y_0) and (x_1, y_1) is given by

$$L = \int_{x_0}^{x_1} ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx$$

We can use Euler's euqation given in 4a with

$$f(y, y', x) = \sqrt{1 + y'^2}$$

Inserted into Euler's equation, we get

$$0 = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) - 0 = 0$$

which is easily integrated to

$$\frac{y'}{\sqrt{1 + y'^2}} = A = \text{const.} \quad \Rightarrow \quad y' = B = \text{const.}$$

Integration one more times gives

$$y = Bx + C$$

which is the equation for a straight line. The constants B and C are given by the condition that the line should pass through (x_0, y_0) and (x_1, y_1).

Problem 5

a) Since we in the b) part want to solve the equations of motion for our new transformed system, we would rather like to choose a transformation such that these are as easy as possible to solve. We can then use Hamilton-Jacobi's equation to find a generating function $S(q, P, t)$ which generates the canonical transformation. Since the Hamiltonian doesn't depend explicitly on time, we can make the Ansatz

$$S(q, P, t) = W(q, P) - E(P)t$$

which gives us Hamilton-Jacobi's characteristic equation

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}kq^2 = E(P)$$

We can either use S or W to generate the transformation. The only difference is that the new Hamiltonian $K = 0$ if S is used and $K = E(P)$ if W is used. We will here use W to generate the transformation. Now choose $E = P = \alpha$ where we know that $P = \alpha =$ constant since the hew Hamiltonian is only a function of the new canonical momenta P (by construction). We then get that W is given by

$$W(q, P) = \pm\sqrt{mk} \int \sqrt{\frac{2P}{k} - q^2 dq}$$

Our transformation (of type B) then gives the connection between old and new variables

$$p = \frac{\partial W}{q} = \pm\sqrt{mk}\sqrt{\frac{2P}{k} - q^2} \tag{6}$$

$$Q = \frac{\partial W}{P} = \sqrt{mk}\frac{2}{k}\frac{1}{2}\int \frac{dq}{\sqrt{\frac{2P}{k} - q^2}} = \sqrt{\frac{m}{k}}\arcsin\left(\frac{q}{\sqrt{\frac{2P}{k}}}\right) \tag{7}$$

From Eq. (7) we can solve for q as a function of Q and P, which, when inserted in Eq. (6) gives p as a function of Q and P. We then get

$$q = \sqrt{\frac{2P}{k}}\sin(\omega Q) \tag{8}$$

$$p = \sqrt{2Pm}\cos(\omega Q) \tag{9}$$

with $\omega = \sqrt{\frac{k}{m}}$.

b) The generating function $W(q, P)$ give the new Hamiltonian

$$K = H = E(P) = P = \alpha$$

Hamilton's canonical equations now give us

$$\begin{aligned} \dot{Q} &= \frac{\partial K}{P} = 1 \\ \dot{P} &= -\frac{\partial K}{Q} = 0 \end{aligned}$$

with the solution

$$\begin{aligned} Q &= t + \beta' \qquad ; \qquad \beta' = \text{ const.} \\ P &= \alpha = \text{ const.} \end{aligned}$$

Inserted into Eqs. (8)–(9) we get the solution

$$\begin{aligned} q(t) &= \sqrt{\frac{2\alpha}{k}}\sin(\omega t + \beta) \qquad ; \qquad \beta = \omega\beta' \\ p(t) &= \sqrt{2\alpha m}\cos(\omega t + \beta) \end{aligned}$$

where (α, β) will be determined from the initial conditions.