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Solutions to

Exam in Analytichal Mechanics, 5p

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Solutions are also available on
http://www.physto.se/~edsjo/teaching/am/index.html.

Problem 1

a) The kinetic energy is given by

$$T = \frac{1}{2}m\left(R\sin\theta\omega\hat{\boldsymbol{\varphi}} + R\dot{\theta}\hat{\boldsymbol{\theta}}\right)^2 = \frac{1}{2}mR^2\left(\omega^2\sin^2\theta + \dot{\theta}^2\right)$$

and the potential energy is given by

$$U = mgR\left(1-\cos\theta\right)$$

The Lagrangian is then given by

$$L = T - U = \frac{1}{2}mR^2\left(\omega^2\sin^2\theta + \dot{\theta}^2\right) - mgR\left(1-\cos\theta\right)$$

and it's derivatives are

$$\begin{cases} \frac{\partial L}{\partial\theta} & = & mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\ \frac{\partial L}{\partial\dot{\theta}} & = & mR^2\dot{\theta} \end{cases}$$

Inserted into Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

we get

$$mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \tag{1}$$

which is the equation of motion for θ.

b) From Eq. (1) we see that $\ddot{\theta} = 0$ for $\theta = 0$ and hence $\theta = 0$ is an equilibrium point. To find if it is stable or not, we Taylor expand the right-hand side in Eq. (1) keeping terms up to linear order in θ, i.e. we set

$$\begin{cases} \sin\theta & \simeq & \theta \\ \cos\theta & \simeq & 1 \end{cases}$$

which gives

$$mR^2\ddot{\theta} \simeq \left(mR^2\omega^2 - mgR\right)\theta$$

This equation has oscillating cos and sin solutions if the coefficient in front of θ in the right-hand side is negative, otherwise the solution is exponentials. Hence, for the solution to be stable, the coefficient has to be negative, i.e.

$$mR^2\omega^2 - mgR < 0$$
$$\Rightarrow \omega^2 < \tfrac{g}{R}$$
$$\Rightarrow \omega_c = \sqrt{\tfrac{g}{R}}$$

c) From Eq. (1) we see that $\ddot{\theta} = 0$ when

$$\sin\theta \left(mR^2\omega^2 \cos\theta - mgR\right) = 0.$$

We see that this equation is fulfilled when

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{g}{R\omega^2}$$

The first of these gives the two equilibrium points $\theta = 0$ and $\theta = \pi$, whereas the second equation only has a solution when $\omega > \omega_c$ and then the equilibrium point is

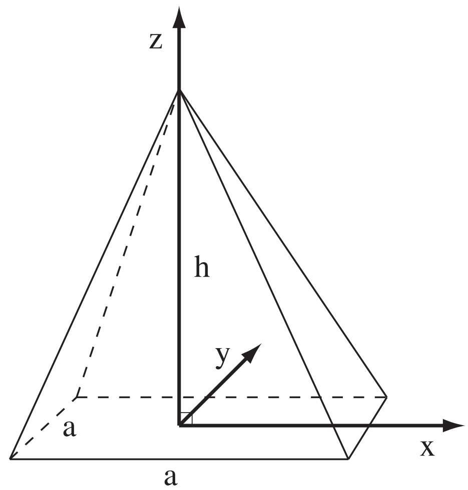
$$\theta = \arccos\frac{g}{R\omega^2}$$

One can show that this equilibrium point is stable by Taylor expand the right-hand side in Eq. (1) around $\theta = \arccos\frac{g}{R\omega^2}$.

Problem 2

a) We start by introducing a cartesian coordinate system with origo in the center of the base and with the x and y axes perpendicular to the sides of the base (se figure).
To calculate the center of mass, we must first calculate the volume to find the density of the pyramid. Note that the volume of a thin layer with height dz on height z is given by

$$dV = dz\left[\left(1 - \frac{z}{h}\right)a\right]^2 = a^2\left(1 + \frac{z^2}{h^2} - 2\frac{z}{h}\right)dz$$



The volume is thus

$$V = \int_0^h a^2\left(1 + \frac{z^2}{h^2} - 2\frac{z}{h}\right)dz = a^2\left[z + \frac{z^3}{3h^2} - 2\frac{z^2}{2h}\right]_0^h = a^2\left[h + \frac{h}{3} - h\right] = \frac{1}{3}a^2h$$

This gives us the density

$$\rho = \frac{m}{V} = \frac{3m}{a^2h}$$

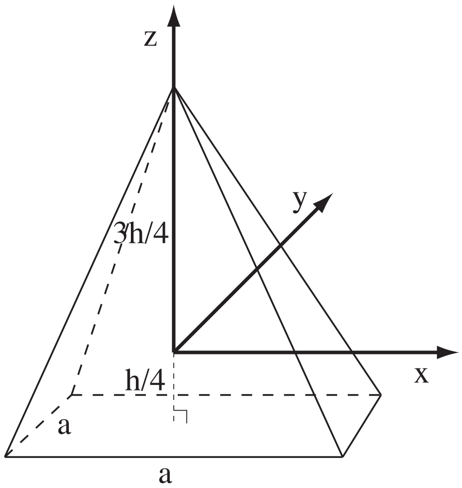
The mass in a thin layer with height dz on height z is thus

$$d\sigma = \rho dV = \frac{3m}{h}\left(1+\frac{z^2}{h^2}-2\frac{z}{h}\right)dz$$

Due to the symmetry, the center of mass S has to lie on the z axis and we can therefore calculate the z coordinate of the center of mass as a weighted mean of z where the weight is the mass fraction in each thin layer, i.e.

$$z_S = \frac{1}{m}\int_0^h z d\sigma = \frac{1}{m}\frac{3m}{h}\int_0^h \left(z+\frac{z^3}{h^2}-2\frac{z^2}{h}\right)dz = \frac{3}{h}\left[\frac{z^2}{2}+\frac{z^4}{4h^2}-2\frac{z^3}{3h}\right]_0^h = \cdots = \frac{h}{4}$$

b) Now introduce a new coordinate system with origo in the center of mass S according to the figure. The components of the tensor of inertia in this system is given by



$$I_{ij} = \int\int\int \rho\, dxdydz \left[\delta_{ij}\mathbf{r}^2 - r_i r_j\right]$$

where $\mathbf{r}$ is the position vector to a point in the pyramid. Since we have chosen our coordinate system such that the pyramid is mirror symmetric in both the xz and yz plane, all products of inertia are zero. As an example, consider the xz component of the tensor of inertia,

$$I_{xz} = \int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dy \rho\, xz$$

where the limits of the various integrations have been explicitly given. The integral in the x direction is an integral of an odd function over a symmetric interval and is thus zero. Hence $I_{xz} = 0$. In the same way, one can show that all products of inertia are zero.

The moments of inertia I_{xx}, I_{yy} and I_{zz} are not zero though, and has to be calculated. Let us start with I_{zz},

$$\begin{aligned}
I_{zz} &= \int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dy \rho \left(x^2+y^2\right) \\
&= \frac{3m}{a^2h}\int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dy \left(x^2+y^2\right) \\
&= \frac{3m}{a^2h}\int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \left[x^2y+\frac{y^3}{3}\right]_{y=-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{y=\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} \\
&= \cdots \quad = \frac{ma^2}{10}
\end{aligned}$$

In the same way we can calculate I_{xx},

$$\begin{aligned}
I_{xx} &= \int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dy\rho \left(y^2+z^2\right) \\
&= \frac{3m}{a^2h} \int_{-\frac{h}{4}}^{\frac{3h}{4}} dz \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dx \int_{-\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)}^{\frac{a}{2}\left(\frac{3}{4}-\frac{z}{h}\right)} dy \left(y^2+z^2\right) \\
&= \cdots \quad = m\left[\frac{a^2}{20}+\frac{3h^2}{80}\right]
\end{aligned}$$

Due to the symmetry, $I_{yy} = I_{xx}$ and we thus have all the components of the tensor of inertia calculated. The tensor of inertia is thus given by

$$\mathbf{I} = \begin{pmatrix} m\left[\frac{a^2}{20}+\frac{3h^2}{80}\right] & 0 & 0 \\ 0 & m\left[\frac{a^2}{20}+\frac{3h^2}{80}\right] & 0 \\ 0 & 0 & \frac{ma^2}{10} \end{pmatrix}$$

Problem 3

We have one degree of freedom and can choose the extension x of the spring AB as compared to its natural length as our generalized coordinate.

a) At equilibrium, $x = x_0$, the force on the mass m from gravitation and the spring must cancel, i.e.

$$mg = kx_0 \qquad \Rightarrow \qquad x_0 = \frac{mg}{k}$$

b) The potential energy can be written as

$$V(x) = -mgx + \frac{1}{2}kx^2$$

The kinetic energy is given by the motion of the mass m and the rotation of the cylinder. Note that the rotation angle of the cylinder is given by x/R and its angular frequency is thus $\omega = \dot{x}/R$. The moment of inertia with respect to the symmetry axis of the cylinder is $I = MR^2$ and the kinetic energy is therefore

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{\dot{x}}{R}\right)^2 MR^2 = \left(\frac{M}{2}+\frac{m}{2}\right)\dot{x}^2.$$

The Lagrangian is now given by

$$L = \left(\frac{M}{2}+\frac{m}{2}\right)\dot{x}^2 + mgx - \frac{1}{2}kx^2$$

The derivatives with respect to $\dot{x}$ and x are given by

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= (M+m)\,\dot{x} \\
\frac{\partial L}{\partial x} &= mg - kx
\end{aligned}$$

Lagrange's equations now yield

$$(M+m)\,\ddot{x}+kx=mg$$

The homogenous equation has the solution

$$x_h = A\cos\left(\sqrt{\frac{k}{M+m}}t+\beta\right)$$

and the particular solution is given by

$$x_p = \frac{mg}{k}.$$

The general solution is thus

$$x = A\cos\left(\sqrt{\frac{k}{M+m}}t+\beta\right)+\frac{mg}{k}.$$

Our initial condition $\dot{x}(0)=0$ gives $\beta=0$ and $x(0)=0$ gives $A=-\frac{mg}{k}$. The solution with our initial conditions is therefore

$$x(t)=\frac{mg}{k}\left[1-\cos\left(\sqrt{\frac{k}{M+m}}t\right)\right]$$

i.e., we have oscillations around our equilibrium point x_0 with the amplitude $\frac{mg}{k}$ and the angular frequency $\sqrt{\frac{k}{M+m}}$.

Problem 4

a) See Scheck, section 2.23, and the lecture notes

b) One can show that $S(\underset{\sim}{q},\underset{\sim}{P},t)$ generates a canonical transformation in two ways:

   i) One can first use Hamilton's variational principle and show that $\Phi(\underset{\sim}{q},\underset{\sim}{Q},t)$ can generate a canonical transformation (and derive the relations between old and new variables) and then derive $S(\underset{\sim}{q},\underset{\sim}{P},t)$ as the Legendre transformation of Φ (apart from the sign) with respect to $\underset{\sim}{Q}$ (see Scheck, section 2.23), or

   ii) one can start from Hamilton's variational principle directly and show that $S(\underset{\sim}{q},\underset{\sim}{P},t)$ can generate a canonical transformation.

Below we will show how to proceed with the proof according to ii).

We can show that the transformation is canoncial by requiring that Hamilton's variational principle is fulfilled in both the old variables,

$$\delta\int\left[\sum_i p_i\dot{q}_i - H(\underset{\sim}{q},\underset{\sim}{p},t)\right]dt=0 \tag{2}$$

and in the new variables,

$$\delta\int\left[\sum_i P_i\dot{Q}_i - \tilde{H}(\underset{\sim}{Q},\underset{\sim}{P},t)\right]dt=0. \tag{3}$$

This is guaranteed if the integrands in Eqs. (2) and (3) do not differ by more than a total time derivative of a function $M(\underset{\sim}{q}, \underset{\sim}{p}, \underset{\sim}{Q}, \underset{\sim}{P}, t)$, i.e. if

$$\sum_i p_i \dot{q}_i - H(\underset{\sim}{q}, \underset{\sim}{p}, t) = \sum_i P_i \dot{Q}_i - \tilde{H}(\underset{\sim}{Q}, \underset{\sim}{P}, t) + \frac{dM}{dt}. \tag{4}$$

Now choose

$$M = S(\underset{\sim}{q}, \underset{\sim}{P}, t) - \sum_i Q_i P_i \tag{5}$$

where we have the right to add the last sum since the equations of motion do not change by adding a total time derivative of the canonical variables. Now differentiate M with respect to t,

$$\frac{dM}{dt} = \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial S}{\partial P_i}\dot{P}_i + \frac{\partial S}{\partial t} - \sum_i \dot{Q}_i P_i - \sum_i Q_i \dot{P}_i$$

If we insert this expression in Eq. (4) and demand that the factors in front of $\dot{q}_i$ and $\dot{P}_i$ cancel (it is because we want to to this identification we subtracted the sum $\sum_i Q_i P_i$ in Eq. (5)) we get the relations between the old and new variables

$$\begin{cases} p_i = \frac{\partial S}{\partial q_i} \\ Q_i = \frac{\partial S}{\partial P_i} \end{cases}$$

For Eq. (4) to be valid, the Hamilton functions must alse satisfy

$$\tilde{H} = H + \frac{\partial S}{\partial t}.$$

Problem 5

a) Noether's theorem looks like this

If the Lagrangian $L(\underset{\sim}{q}, \dot{\underset{\sim}{q}})$ describes an autonomous system which is invariant under the transformation $\underset{\sim}{q} \to \underset{\sim}{h}^s(\underset{\sim}{q})$ where s is a real continuous parameter such that $\underset{\sim}{h}^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ is the identity transformation, then

$$I(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s(\underset{\sim}{q}) \Bigg|_{s=0}$$

is a constant of motion.

and it is proven e.g. like this:

Let $\underset{\sim}{q} = \underset{\sim}{\varphi}$ be a solution to Lagrange's equations. Since the system is invariant under the transformation $\underset{\sim}{h}^s$

$$\underset{\sim}{q}(s,t) = \phi(s,t) = \underset{\sim}{h}^s(\underset{\sim}{\varphi}(t))$$

is also a solution to Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)\right) = \frac{\partial L}{\partial q_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) \tag{6}$$

Further on, L, is invariant under the transformation, i.e.

$$0 = \frac{d}{ds} L(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) = \sum_{i=1}^{f} \left[\frac{\partial L}{\partial q_i} \frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{\phi}_i}{ds} \right] \tag{7}$$

Now use Eq. (6) to replace $\frac{\partial L}{\partial q_i}$ in Eq. (7) and change the order of the derivates in the last term. We then get

$$\sum_{i=1}^{f} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left(\frac{d\phi_i}{ds} \right) \right] = 0$$

This must especially hold when $s = 0$ and we then finally get (with $d\phi_i/ds = dh_i^s/ds$)

$$\frac{d}{dt} \left(\sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{dh_i^s}{ds} \right) \Bigg|_{s=0} = 0 \qquad \Rightarrow \qquad \frac{d}{dt} I = 0$$

b) The Lagrangian given is invariant under rotations around an arbitrary axis. Let us put our coordinate system such that this arbitrary axis is the z axis. We can then write the transformation as

$$h^s : \quad \mathbf{r} = (x, y, z) \rightarrow \mathbf{r}' = (x', y', z') = (x \cos s + y \sin s, -x \sin s + y \cos s, z)$$

We then get

$$\frac{d}{ds} h^s \Bigg|_{s=0} = (y, -x, 0) = \mathbf{r} \times \hat{\mathbf{z}}$$

Our constant of motion is then, according to Noether's theorem,

$$I = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s \Bigg|_{s=0} = m\dot{\mathbf{r}} \cdot \mathbf{r} \times \hat{\mathbf{z}} = \{\text{vektoranalys}\} = \hat{\mathbf{z}} \cdot (m\dot{\mathbf{r}} \times \mathbf{r}) = -\hat{\mathbf{z}} \cdot \mathbf{L} = -L_z$$

i.e. invariance under rotation around the z axis means that the z component of the angular momentum $\mathbf{L}$ is conserved. Since we now have invariance under rotation around any axis, all components of the angular momentum $\mathbf{L}$ must be conserved, i.e. $\mathbf{L}$ is a constant of motion.